# GCSE Mathematics revision notes

# (Foundation).

RM

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# General information

- You must know all your times tables and number bonds to 100.

- Note: some "asides" have been included to make the topics more interesting. These aren't needed for GCSE.

# Contents

Times tables

| One times table | Two times table | Three times table | Four times table |
|---|---|---|---|
| 1 x 1 = 1 | 1 x 2 = 2 | 1 x 3 = 3 | 1 x 4 = 4 |
| 2 x 1 = 2 | 2 x 2 = 4 | 2 x 3 = 6 | 2 x 4 = 8 |
| 3 x 1 = 3 | 3 x 2 = 3 | 3 x 3 = 9 | 3 x 4 = 12 |
| 4 x 1 = 4 | 4 x 2 = 8 | 4 x 3 = 12 | 4 x 4 = 16 |
| 5 x 1 = 5 | 5 x 2 = 10 | 5 x 3 = 15 | 5 x 4 = 20 |
| 6 x 1 = 6 | 6 x 2 = 12 | 6 x 3 = 18 | 6 x 4 = 24 |
| 7 x 1 = 7 | 7 x 2 = 14 | 7 x 3 = 21 | 7 x 4 = 28 |
| 8 x 1 = 8 | 8 x 2 = 16 | 8 x 3 = 24 | 8 x 4 = 32 |
| 9 x 1 = 9 | 9 x 2 = 18 | 9 x 3 = 27 | 9 x 4 = 36 |
| 10 x 1 = 10 | 10 x 2 = 20 | 10 x 3 = 30 | 10 x 4 = 40 |
| 11 x 1 = 11 | 11 x 2 = 22 | 11 x 3 = 33 | 11 x 4 = 44 |
| 12 x 1 = 12 | 12 x 2 = 24 | 12 x 3 = 36 | 12 x 4 = 48 |

| Five times table | Six times table | Seven times table | Eight times table |
|---|---|---|---|
| 1 x 5 = 5 | 1 x 6 = 6 | 1 x 7 = 7 | 1 x 8 = 8 |
| 2 x 5 = 10 | 2 x 6 = 12 | 2 x 7 = 14 | 2 x 8 = 16 |
| 3 x 5 = 15 | 3 x 6 = 18 | 3 x 7 = 21 | 3 x 8 = 24 |
| 4 x 5 = 20 | 4 x 6 = 24 | 4 x 7 = 28 | 4 x 8 = 32 |
| 5 x 5 = 25 | 5 x 6 = 30 | 5 x 7 = 35 | 5 x 8 = 40 |
| 6 x 5 = 30 | 6 x6 = 36 | 6 x 7 = 42 | 6 x 8 = 48 |
| 7 x 5 = 35 | 7 x 6 = 42 | 7 x 7 = 49 | 7 x 8 = 56 |
| 8 x 5 = 40 | 8 x 6 = 48 | 8 x 7 = 56 | 8 x 8 = 64 |
| 9 x 5 = 45 | 9 x 6 = 54 | 9 x 7 = 63 | 9 x 8 = 72 |
| 10 x 5 = 50 | 10 x 6 = 60 | 10 x 7 = 70 | 10 x 8 = 80 |
| 11 x 5 = 55 | 11 x 6 = 66 | 11 x 7 = 77 | 11 x 8 = 88 |
| 12 x 5 = 60 | 12 x 6 = 72 | 12 x 7 = 84 | 12 x 8 = 96 |

| Nine times table | Ten times table | Eleven times table | Twelve time table |
|---|---|---|---|
| 1 x 9 = 9 | 1 x 10 = 10 | 1 x 11 = 11 | 1 x 12 = 12 |
| 2 x 9 = 18 | 2 x 10 = 20 | 2 x 11 = 22 | 2 x 12 = 24 |
| 3 x 9 = 27 | 3 x 10 = 30 | 3 x 11 = 33 | 3 x 12 = 36 |
| 4 x 9 = 36 | 4 x 10 = 40 | 4 x 11 = 44 | 4 x 12 = 48 |
| 5 x 9 = 45 | 5 x 10 = 50 | 5 x 11 = 55 | 5 x 12 = 60 |
| 6 x 9 = 54 | 6 x 10 = 60 | 6 x 11 = 66 | 6 x 12 = 72 |
| 7 x 9 = 63 | 7 x 10 = 70 | 7 x 11 = 77 | 7 x 12 = 84 |
| 8 x 9 = 72 | 8 x 10 = 80 | 8 x 11 = 88 | 8 x 12 = 96 |
| 9 x 9 = 81 | 9 x 10 = 90 | 9 x 11 = 99 | 9 x 12 = 108 |
| 10 x 9 = 90 | 10 x 10 = 100 | 10 x 11 = 110 | 10 x 12 = 120 |
| 11 x 9 = 99 | 11 x 10 = 110 | 11 x 11 = 121 | 11 x 12 = 132 |
| 12 x 9 = 108 | 12 x 10 = 120 | 12 x 11 = 132 | 12 x 12 = 144 |

# Chapter 1: numbers and arithmetic
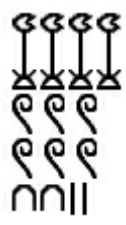
## *What numbers mean*

### Place value

Our number system is a "place value" system where the amount each digit contributes to the total depends on its position relative to the decimal point (or the right-hand end of the number, if it has no point).

> *Aside*: To appreciate how amazingly marvellous this is, remember that numbers have not always been like this. The Egyptians used special symbols for each number

| | | |
|---|---|---|
| I | 1 | |
| ∩ | 10 | Cattle yoke |
| ℘ | 100 | Rope |
| ⚘ | 1000 | Water lily |
| ∫ | 10,000 | Finger |
| 🐸 | 100,000 | Frog |
| 𓀀 | 1 million | Man |

= 4622

For us, though, the digits in each column represent an amount 10× larger than an equivalent digit on its right (or one tenth of the same digit to its left).

So the number 432 means 400 + 30 +2.

The number 4635.179 means 4000 + 600 + 30 + 5 + 0.1 + 0.07 + 0.009

Numbers that we can use for counting a number of items are known as "whole numbers" or "integers". eg -5, 0, 2, 23789

## Fractions

Values in-between whole numbers are usually written as fractions.

A fraction is written as $\dfrac{\text{numerator}}{\text{denominator}}$

(you can call these "top" and "bottom" if you like).

What do they mean?   Imagine a cake cut into 10 equally-sized pieces, I help myself to one.

Each piece is "one tenth" of a cake and we write this as $\dfrac{1}{10}$.   The "10" is the number of bits it is cut into and the "1" is how many of them I get to eat.

If Fred takes three of these "tenths", we write his fraction as $\dfrac{3}{10}$ ("three tenths").   So the number on the bottom tells me how finely the cake is sliced and the number on top is how many slices I get.

A fraction is really the result of a division calculation eg:

- One cake cut into five pieces, each piece is a fifth of a cake could be written as $1 \div 5 = \dfrac{1}{5}$

- Similarly if I have three times as much cake (taking 3 slices, or starting with 3 cakes and taking a fifth of each) : $3 \div 5 = \dfrac{3}{5}$

## Equivalent fractions

Apart from the cake getting a bit crumbly, it doesn't much matter to me whether the cake is cut into 5 pieces of which I eat 2  (so $\dfrac{2}{5}$ of a cake) or 10 pieces of which I eat 4 (so $\dfrac{4}{10}$ of a cake) or 20 pieces of which I eat 8 (so $\dfrac{8}{20}$ of a cake).  These are all equivalent fractions.

## Improper ("top-heavy") fractions

If the number on the top is bigger than the denominator, the number is greater than one.  So if there are several cakes, each cut into 4 pieces, and (being *very* greedy) I take 6 pieces I have $\dfrac{6}{4}$ cake which is the same as "one and a half" cakes.

## Decimals

Decimals are just a quick way of writing any fraction with a power of 10 (10, 100, 1000 etc) as the denominator.
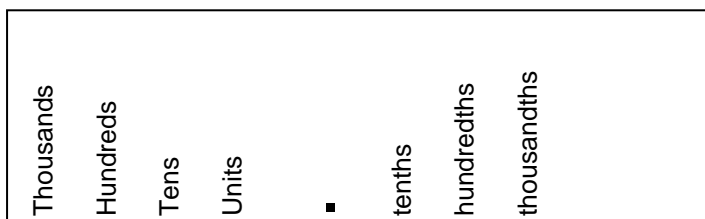


0.3 means 3 tenths = $\dfrac{3}{10}$

0.05 means 5 hundredths = $\dfrac{5}{100}$

0.009 means 9 thousandths = $\dfrac{9}{1000}$

0.359 means 3 tenths + 5 hundredths + 9 thousandths = $\dfrac{3}{10} + \dfrac{5}{100} + \dfrac{9}{1000}$

Now then!  Since $\dfrac{1}{10} = \dfrac{10}{100} = \dfrac{100}{1000}$, one could also write this as

$$\dfrac{300}{1000} + \dfrac{50}{1000} + \dfrac{9}{1000} = \dfrac{359}{1000}$$

This means that adding a 0 to the end does not change the value:

$0.3590 = \dfrac{3590}{10000} = \dfrac{359}{1000} = 0.359$  (but it does often imply that it may have been rounded from a value between 0.35895 and 0.35905 as opposed to between 0.3585 and 0.3595 so it is "more accurate").

So 0.15 is a number half-way between 0.1 and 0.2 (think of 15 cm = 0.15 m, halfway between 10 cm and 20 cm on a ruler, or  £0.15 being halfway between £0.10 and £0.20).

> ➤ *If you understand fractions, you understand decimals too!*

$$\dfrac{3}{10} = 0.3$$

> ➤ *If you understand metres and centimetres, you understand decimals too!*

I am 176 cm tall which is $\dfrac{176}{100} = 1.76$ metres

> ➤ *If you understand pounds and pence, you understand decimals too!*

A goldfish costs 405p which is $\dfrac{405}{100} = £4.05$

### *Fractions that produce recurrent decimals*

- Fractions where the denominator divides exactly into a power of 10 have exact decimal equivalents.

  o For instance, to find $\dfrac{1}{4}$ as a decimal we think that 100÷4 = 25 (4 divides into 100, which is a power of 10) so we can write $\dfrac{1}{4} = \dfrac{25}{100} = 0.25$.

- Fractions where the denominator is a multiple of any prime number (other than 2 and 5) do not have such simple decimal equivalents.

  o $\dfrac{1}{3} = 0.3333\dot{3}$ where the $\dot{}$ shows that the decimal repeats ad infinitem (hence $\dfrac{2}{3} = 0.6666\dot{6}$ and $\dfrac{3}{3} = 0.9999\dot{9}$ which rounds to = 1, regardless of how many decimal places you use). **Must remember 1/3 = 0.33333 = 33.33%**

  o $\dfrac{1}{7} = 0.\dot{1}42857\dot{7}$ (the double dots indicate that the 142857 repeats, so 0.142857142857 etc)

  o $\dfrac{1}{11} = 0.\dot{0}\dot{9}$ (so it repeats 0.0909090909 etc)

## Comparing numbers

### Comparing two positive numbers

Write the two numbers so their decimal points line up. Starting at the left side, compare the pairs of numbers in each column until you reach a column where one value is bigger than another.

### Examples

To compare 1276.5 and 901.03:
```
1276.5
 901.03
```
In the left hand column, the top number has a "1" and the lower number has no digit, so the top number is bigger

To compare 1.54329 and 1.5671:
```
1.54329
1.5671
```
In the third column, the top number has a "4" and the bottom number has a "6" so the second number is bigger.

## Percentages

A percentage is simply another way of writing a fraction or decimal. "Percentage" means a fraction with 100 as the denominator (number of pieces of cake with the cake cut into 100 slices).

To convert any number, fraction or decimal to a percentage, multiply by "100%".

This is effectively multiplying by 1 (since 100% = 1) - you are multiplying by 100, then the % sign makes everything 100 times smaller again.

eg $\dfrac{1}{4} = \dfrac{1}{4} \times 100\% = 25\%$

This is particularly easy if starting with decimals - just more the decimal point 2 places to multiply by 100, eg

$0.359 = 0.359 \times 100\% = 35.9\%$

Going the other way, remove the % sign and divide by 100:

$25\% = 25 \div 100 = \dfrac{1}{4}$

Examples:

- "25% of 200" means 25÷100 x 200 = 25 x 2 = 50

- "Write 80 out of 2000 as a percentage" means write 80÷2000 x 100% = 80÷20 x 1% = 4%

# Negative numbers

As well as using numbers for counting, we can use them to indicate how much something has "changed" from a previous value.

So if I have £10 in my bank account and add another £50, I then have £60.

If I have £10 in the account and take out £50 (so I'm overdrawn), what is a sensible way of writing this? I say I have "minus" £40 in the account, which is -£40, a negative amount.

Usually you see negative numbers as temperatures:

- On a hot day the air temperature might be $25°C$

- Water freezes at $0°C$.

- A very cold and frosty night might be $-10°C$

- If you go up Kilimanjaro, it is $-25°C$ and you know it.

```
−7 −6 −5 −4 −3 −2 −1   1  2  3  4  5  6  7  8  9 10
```
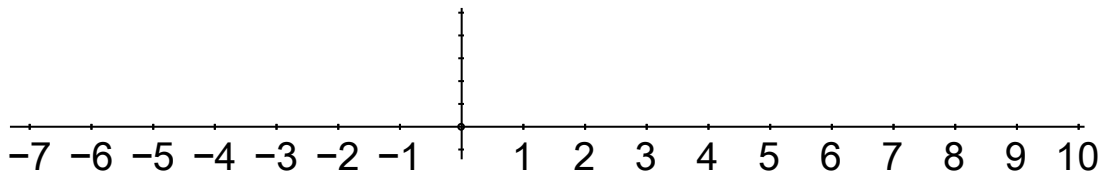
## Adding and subtracting negative numbers

➢ To add a positive number, move to the right → on the number line

  ○ To subtract a positive number, move to the left ← on the number line

➢ To add a negative number, move to the left ← on the number line

 [ think, 5+(-2) is the same as 5-2=3 ]

  ○ To subtract a negative number, move to the right → on the number line

   [Think, 5-(-2) is the same as 5+2=7 ]

## Multiplying and dividing by negative numbers

Every time we multiply or divide by a negative number, we change the sign (positive or negative) of our answer.

➢ if your multiplication contains an even number of –signs, your answer is positive

➢ if your multiplication contains an odd number of –signs, your answer is negative

Examples:

$$2 \times -3 = -6$$

$$2 \times -3 \times -4 = -24$$

$$\frac{-2}{-4} = \frac{1}{2}$$

## Standard form (now in the Higher Paper only)

We tend to write very large or very small numbers with a "powers of ten" multiplier instead of umpteen zeros.

So $1250 = 1.25 \times 10^3$ (where $10^3 = 10 \times 10 \times 10 = 1000$). The "3" means "move the decimal point 3 places to the right, so 1.25 become 1250, then lose the $10^3$ multiplier".

Similarly 0.019 becomes $1.9 \times 10^{-2}$ (where $10^{-2} = 0.01$). The rule is that the first number ends up being between 1 and 10.

Remember your powers of 10:

| $10^3$ | $10^2$ | $10^1$ | $10^0$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ |
|--------|--------|--------|--------|-----------|-----------|-----------|
| 1000 | 100 | 10 | 1 | 0.1 | 0.01 | 0.001 |

Measurements with metric units are very easy to change into standard form e.g.

5 kg = $5 \times 10^3$ g

7 cm = $7 \times 10^{-2}$ m

Note: A number in standard form should be written with the number part $\geq 1$ but <10. For instance,

$1.25 \times 10^3$, $12.5 \times 10^2$ and $125 \times 10^1$ are all = 1250 but only the first gets you the marks for a "standard form" answer.

*See "Further Algebra" for laws of indices and how to add, multiply and divide standard form numbers.*

12

# *Arithmetic – how to do things with numbers*

## Addition

Write it down! I know you can do sums in your head but you are making life hard for yourself. It is quicker to use pen and paper and it means:

> ➤ you can go back and check it at the end of the exam

> ➤ the examiner can give you marks for working even if the final answer comes out wrong

Write the numbers so the decimal points line up (or the units column if no point). Add the right hand column, then the one to its left etc.

```
1234
 325 +
1559
```

*How to carry:*

When a column adds to more than 9, write its units digit and carry the tens digit as a little number in the next column to be added later, eg

```
 4785
 6517 +
   ₁2
```

(think 5+7=12). Now add the next column, 8+1+1=10 so write 0 and carry 1.

```
 4785
 6517 +
  ₁02
```

The next column has 7+5+1=13, write 3, carry 1.

```
 4785
 6517 +
 ₁302
```

Finally 4+6+1=11, write 1, carry 1 but as there are no numbers left to add this becomes a proper 1 in the answer:

```
 4785
 6517 +
11302
```

Check you can add 3 or more numbers and/or decimals in the same way, eg

```
 132              1.23
 405              4.107 +
 789 +            5.337
1326
```

## Subtraction

You need to know "number bonds" to 20 so you can immediately see the answer to sums like 16-9 or 11-7.

For a subtraction like 96-31, write as

```
96
31  −
65
```

*"Borrowing" when subtracting a large digit from a small one*

Sometimes we need to add 10 to a digit to help us subtract.  We do this by borrowing a "1" from the next column to the left.  So we don't forget we have done this, we cross out the original number and write the new version above, eg: to do 742-259

Cannot take 9 from 2, so borrow 1 from the 4 (it becomes 3) and take 9 from 12:

```
742
259  −
  3
```

Now cannot take 5 from 3 so borrow 1, take 5 from 13

```
742
259  −
 83
```

Finally take 2 from 6:

```
742
259  −
483
```

We can check our answer: if we add 259+483 we should get back to 742.

<u>Borrowing from further left</u>

Sometimes we have to borrow from beyond the next column.

To find 6005 – 787 I have to borrow 10 from the 6000.  The 10 makes the 5 up to 15, so I can subtract 7, leaving 5990 behind in the other columns:

```
6005
 787  -
5218
```

*Subtracting a large number from a small one*

If we subtract a large number from a small one, the answer will be a negative number.  The easiest way is to subtract them the other way around, so the answer is positive, then make your final answer negative eg:

To find 259-742, show working for 742-259 = 483 as above, then write "hence 259-742 = -483".

## Adding lists of positive and negative numbers

Add up all the positive numbers, add the negative numbers, then combine the answers eg:

"Find the sum of these numbers:

```
3  4  -12  18  -33  20  -5 "
```

We think 3+4+18+20 = 7+18+20 = 25+20 = 45 and  -12 -33 -5 = -45-5 = -50

Then 45-50 = -5.

## Multiplication

Again, we write the two numbers so the units columns line up vertically.   It doesn't really matter which is the top number but it is easiest to put the number with the most non-zero digits on the top line.

To multiply 23×3, think 3×3=9,

```
 23
  3  ×
 ──
  9
```

and then 2×3=6,

```
 23
  3  ×
 ──
 69
```

### *Carrying*

As with addition, if when we multiply digits the answer is >9 we must carry the tens digit into the next column.  We write it small so we don't forget it.

Example:   57×7, first multiply the right-hand digit in the top number, 7×7=49

```
 57
  7  ×
 ──
 ₄9
```

Now multiply the next digit, 5×7=35, 35+4=39

```
  57
   7  ×
 ───
 399
```

### *Multiplying longer numbers*

Put the number with the most non-zero digits on the top line.  Each digit in the second number will generate its own line as part of the solution.

**Example:**   "Multiply 23×16"

We can (as above) easily do 23×6=138 and 23×10=230 so we write two intermediate answers, then add them to get the final answer

```
  23
  16  ×
 ───
 138
 230  +
 ───
 368
```

**Note how when we multiply 23×10, the 23 shifts to the left by 1 column because the 1 is in the tens column, one place to the left of the units column.**

**Example**

"Multiply 234×321"

This time we will have 3 intermediate answers (234×1, 234×20, 234×300) to add up

```
  234
  321  ×
  234
 4680
70200  +
75114
```

*Multiplying by powers of 10*

To multiply by any power of 10, move the decimal point to the right or left.

**Examples**

$12.3×1000 = 12300.$  (multiplying by $10^3$, the decimal point moves 3 places to the right)

$9.71×0.01 = 0.0971$  (multiplying by $10^{-2}$, the decimal point moves 2 places to the left)

*Multiplying decimal numbers*

Just the same, but your answers move to the right when multiplying by digits to the right of the units column (or use the lattice method as above).

**Example**

"Multiply 2.4×0.12"

```
2.4
0.12  ×
0.24          2.4 moves 1 place to right when multiplied by 0.1
0.048  +      4×2=8, move 2 places to right as the 2 is two to right of the units column etc
0.288
```

# The lattice method for multiplication

This is another way of multiplying.

- Write the digits along the top and right sides of a grid, splitting each cell in the grid diagonally. Decimal points go on a grid line.

- Multiply each pair of digits, splitting the answer between left and right sides of the cell (if the answer <10, put a 0 in the left side).

- Now add up along each diagonal

- Put a decimal point where the two decimal point grid lines intersect, then follow this down along a diagonal line to find the decimal point position in the answer.

- Add up the digits between the diagonals, <u>starting at bottom right and carrying as appropriate.</u>

- Read off the answer.

**Example**

"Multiply 234×329"



Here 7+3+8=18, write 8, carry (1) to the left.

8+2+6+2 + carried 1 = 19, write 9, carry (1) to left.

1+4+9+1+ carried 1 = 16, write 6, carry 1 to left

6+ carried 1 = 7.

The answer is 76986.

**Example**

"Multiply 2.34×3.29"       Note the lines showing the decimal point position.  The answer is 7.6986

| | 2 . | 3 | 4 | |
|---|---|---|---|---|
| | 0 ╱ 6 | 0 ╱ 9 | 1 ╱ 2 | **3** . |
| 7 ╱ (1) 6 | 0 ╱ 4 | 0 ╱ 6 | 0 ╱ 8 | **2** |
| | 1 ╱ 8 | 2 ╱ 7 | 3 ╱ 6 | **9** |
| (1) 9 | (1) 8 | 6 | | |

## Division

<u>What does division do?</u>

People find it helpful to think of division in two ways

- As a sharing process. 48÷12=4 so if I share £48 between 12 people they get £4 each (answer is a quantity of something)

- As a container-filling process. If I have 48 pints of beer and I fill jugs that hold 4 pints each, how many jugs do I need? 48÷4=12 jugs (answer is a number of objects).

    - I can also say this as "How many 4s go into 48?"

**The order matters.**
- 48 ÷ 4 =12   (we know 4 goes into 48 twelve times)
- $4 \div 48 = \dfrac{1}{12}$   which is not at all the same thing (only 1/12 of 48 goes into 4).

**Dividing by a fraction has the same meaning as dividing by a number**

10 ÷ 5 =2   means how many fives go into ten?  Answer 2

   (so to make £10 we could use two £5 notes)

$6 \div \frac{1}{2} = 12$  means how many halves go into 6?  Answer 12

   (so to make £6 we could use 12  50p coins)

**Dividing by a number is the same as multiplying by the fraction** $\dfrac{1}{\text{number}}$.

$$\text{so }\; 10 \div 5 = 10 \times (1 \div 5) = 10 \times \frac{1}{5} = \frac{10}{5} = 2$$

*<u>How to do it.</u>*

(1) For simple sums we can use mental arithmetic tricks

**Example:** find 12÷400

   Think: 400=4×100 so we can divide by 4 and then divide by 100.

   12÷4=3 and then 3÷100=0.03

(2) When dividing by small numbers, sometimes the answer is obvious eg

24680÷2=12340   (just halving each digit)

(3) If the answer is not obvious you need either <u>short</u> or <u>long division</u>.
- ➢ Long division always works
- ➢ Short division may be easier when dividing by a single digit number (2 to 9). For longer numbers it gets messy and long division is easier to write out.

## Short division

**Example** "Find 7854÷3"
First we swap the order, so the 3 is on the left:

$$3\overline{)7\ 854}$$

Now we divide 7÷3 to get 2 and write this above the 7 as part of the answer.
2×3=6 not 7, there is 1 left so we write this in front of the 8 to make an 18:

$$3\overline{)7^1854}\quad^2$$

Now we do 18÷3 = 6, write this as the next part of the answer: (No remainder to carry).

$$3\overline{)7^1854}\quad^{2\ 6}$$

5÷3 = 1 remainder 2, write the 1 above, put a small 2 in front of the 4 to make 24:

$$3\overline{)7^185^24}\quad^{2\ 61}$$

Finally, 24÷3 = 8:

$$3\overline{)7^185^24}\quad^{2\ 61\ 8}$$

*Answer 2618.*

**Example**
Find 59.5 ÷ 0.7

We would prefer to divide by 7, not 0.7, so first "balance" the calculation by multiplying each side by 10 hence 595 ÷ 7

5 does not divide by 7 but we know 7×8=56 so 56÷7 = 8. Then 59÷7 = 8 remainder 3:

$$7\overline{)59^35}\quad^8$$

Now we do 35÷7 = 5:

$$7\overline{)59^35}\quad^{8\ 5}$$

*Answer 85.*

## Long division

The principle is just the same as for short division but we write out the subtractions in rows to make it easier when using multi-digit numbers.

**Example** "Find 157.5÷7"

First we swap the order, so the 7 is on the left:

$$\begin{array}{r} 22.5 \\ \hline 7)\overline{157.5} \end{array}$$

$$\underline{140\ -}$$      Think $20\times 7=140$, put 2 in tens column of answer, write 140 to subtract

$$17.5$$

$$\underline{14\ -}$$      Think $2\times 7=14$, put 2 in units column of answer, write 14 to subtract

$$3.5$$

$$\underline{3.5-}$$      Think $0.5\times 7=3.5$, put 5 in tenths column of answer, write 3.5 to subtract

$$0$$

    *Answer 22.5*

**Example** "Find 5301÷31"

First we swap the order, so the 31 is on the left:

$$\begin{array}{r} 171 \\ \hline 31)\overline{5301} \end{array}$$

$$\underline{3100\ -}$$      Think $100\times 31=3100$, put 1 in hundreds column of answer, write 3100 to subtract

$$2201$$

$$\underline{2170\ -}$$      Think $70\times 31=2170$, put 7 in tens column of answer, write 2170 to subtract

$$31$$

$$\underline{31-}$$      Think $1\times 31=31$, put 1 in units column of answer, write 31 to subtract

$$0$$      Remainder 0 so we've finished.

    *Answer 171*

## Squares and cubes

To square a number we multiply it by itself, $4^2=4\times4=16$

To cube, we multiply it itself again: $4^3=4\times4\times4=64$

Remember:

- Multiplication is a simple way of writing repeated addition, can write 3+3+3+3+3 as 3×5

- Indices are a simple way of writing repeated multiplication, can write 3×3×3×3×3 as $3^5$

On a calculator, use the $[X^2]$ and $[X^3]$ buttons e.g. to square 14.92, do

14.92 $[X^2]$ =

| X | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|
| $X^2$ | 1 | 4 | 9 | 16 | | | | | | | | |
| $X^3$ | 1 | 8 | 27 | 64 | | | | | | | | |

(fill in yourself). You need to know these so you can spot them in a list.

You can also use the [^] calculator button to get any power of a number e.g. to calculate $3^5$ do:

3 [^] 5 =

(just like the +, -, × and ÷ symbols, ^ comes between the two numbers).

## Square and cube roots

*Square rooting converts a square back into the original number.*

You may be asked to spot the square roots of numbers like 4, 9, 16, 25, 36 etc (see table above).

There are ways of finding them using pen and paper but you don't need to know this

Using a calculator, use the [√] button e.g.

[√] 53361 =

Tip:

- Buttons that contain X such as the $[X^2]$ and $[X^3]$ above are pressed after you enter the number.

- Other buttons like [√] come before the number, just like writing √53361 on paper.

# Prime numbers

A prime number only divides (to give an integer answer) by itself and 1.

- 1 is defined as not being a prime.

- All prime numbers are odd except 2.

They go:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 73, 79, 83, 89, 97….

(all the numbers that cannot occur as answers in the times tables).  You need to be able to spot them in a list.

Aside:  to check if a number is prime:

- Is it odd (or = 2)?

- Does it divide by 3 (digits would add to a multiple of 3)

- Does it divide nicely by any odd number (don't need to check above its square root).

**Example:**  is 18241 a prime number?

- It's not even

- 1+8+2+4+1=16, 1+6=7 so it is not divisible by 3

- Get a calculator, try dividing by 5 (no), 7 (no), 11 (no), 13 (no), 17 (Ah! 18241÷17=1073).   So it's not a prime.

# Factors and multiples

Example:

- The factors of 24 are 2, 3, 4, 6, 8 and 12 since we can write 24 as 2×12, 3×8 and 4×6

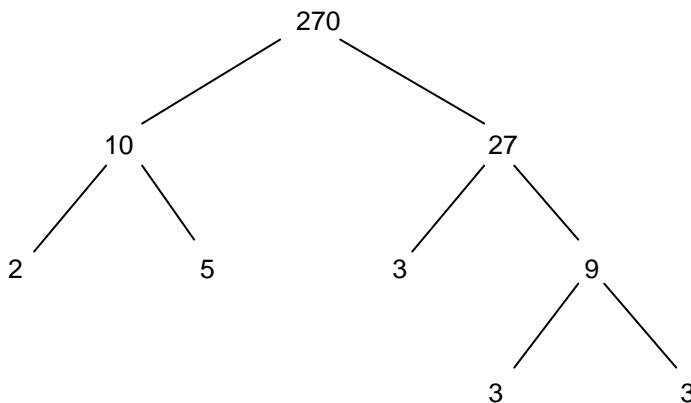- 12, 18, 24, 30 36 etc are multiples of 6

There are 3 things we may want to do:

- Find all the factors of a number

- Find the Highest Common Factor (HCF) of two numbers, eg the HCF of 12 and 18 is 6 (they both divide by 6)

- Find the Lowest Common Multiple (LCM) of two numbers, e.g. the LCM of 12 and 18 is 72 (72 appears in the lists of multiples for both 12 and 18)

*To find the prime factors of a number*

Make a factor tree eg for 270

```
                        270
                       /    \
                     10      27
                    /  \    /  \
                   2    5  3    9
                              /  \
                             3    3
```

So we can write $270 = 2 \times 5 \times 3 \times 3 \times 3 = 2 \times 3^3 \times 5$

*To find the highest common factor of a pair of numbers*

- Find the prime factors for each

- Take the lowest power of each prime factor that occurs in both lists

**Example**

Find the HCF of 270 and 252

We know from above that $270 = 2 \times 3^3 \times 5$ and you can make your own factor tree to show that

$252 = 2^2 \times 3^2 \times 7$

The highest common factor is then $2 \times 3^2 = 18$.   (One list only has $2^1$, one only has $3^2$, 5 does not occur in both, 7 does not occur in both).


Alternatively, write out complete factor lists for each:

- 270 has factors 2 × 135, 3 × 90, 5 × 54, 6 × 45, 9 × 30, 10 × 27, 15 × 18

  Putting these in order gives 2, 3, 5, 6, 9, 10, 15, 18, 27, 45, 54, 90, 135

- 252 has factors 2 ×126, 3 × 84, 4 × 63, 6 × 42, 7 × 36, 9 × 28, 12 × 21, 14 × 18.

  Putting these in order:  2, 3, 4, 6, 7, 9, 12, 14, 18, 21, 28, 36, 42, 63, 84, 126

We see that 18 is the largest number that occurs in <u>both</u> lists.

We can use the HCF to find the LCM (below).

_To find the Lowest Common Multiple of a pair of numbers, e.g. 15 and 25_

Either:

Write out lists of multiples for each and pick the lowest that occurs in both lists.

- 15 has multiples 30, 45, 60, 75, 90, 105 etc

- 25 has multiples 50, 75, 100, 125 etc

- The lowest that occurs in both lists is 75

Or:

Multiply the two numbers and divide by the highest common factor.

- 15 = 3 × 5

- 25 = 5 × 5 and we see the highest common factor is 5.

- Therefore the LCM is (3 × 5) × (5 × 5) ÷ 5 = 3 × 5 × 5 = 75

We use lowest common multiples to find a common denominator whenever we add or subtract fractions.

# BIDMAS

BIDMAS helps us remember the rules for evaluating a list of calculations.  We need to know whether a sum written as 1+2×3 is meant to give the answer 9 (adding the 1 and 2 first) or 7 (multiplying the 2 and 3 first).

|   |   |
|---|---|
| B | brackets |
| I | indices (squares, cubes, square roots etc) |
| DM | division and multiplication, working from left to right |
| AS | addition and subtraction, working from left to right. |

Examples

$1+2\times3$         do the multiplication first (2×3=6), then the addition (1+6=7)

$1+3^2$         do the squaring first ($3^2$=9), then the addition (9+1 = 10)

$2\times8\div4$         left to right, 2×8=16, 16÷4=4

$1+4-3+5$         left to right, 1+4=5, 5-3=2, 2+5=7

$(1+2)\times3$         do the brackets first  (1+2=3), then multiplication (3×3=9)

# More arithmetic – how to do things with fractions

## Equivalence and simplification

A fraction does not change in value if the top ("numerator") and bottom ("denominator") are each multiplied by the same value:

$$\frac{2}{3} = \frac{4}{6} = \frac{8}{12}$$ (multiplying top and bottom by 2 each time)

Also true if top and bottom are divided by the same value: $$\frac{3}{9} = \frac{1}{3}$$

> nb:  Under no circumstances should you add anything to top and bottom of a fraction – this changes its value so whatever relationship you started with would no longer be true.  Think
>
> $$\frac{2}{3} \neq \frac{1}{2} \neq \frac{0}{1}$$   ($\neq$ means "not equal to").

## Converting mixed numbers to improper fractions and vice-versa.

To convert mixed to improper, write the integer part as a fraction (so both denominators are the same), then add:

$$3\frac{1}{4} = \frac{3 \times 4}{4} + \frac{1}{4} = \frac{12 + 1}{4} = \frac{13}{4}$$

To convert an improper fraction to a mixed number, write the numerator (top) as a multiple of the denominator + a remainder, eg:

$$\frac{22}{3} = \frac{21 + 1}{3} = \frac{7 \times 3 + 1}{3} = 7\frac{1}{3}$$   (think "*how many times does 3 go into 22?   - 7 times, remainder 1*")

## Addition

**Same denominator** – just add the numerators ("tops")  and integer parts, if any.

$$\frac{1}{5} + \frac{2}{5} = \frac{1 + 2}{5} = \frac{3}{5}$$

$$2\frac{1}{5} + 3\frac{2}{5} = 5 + \frac{1 + 2}{5} = 5\frac{3}{5}$$

**Addition – different denominators.**

Find the lowest common multiple (LCM) of the two denominators and express each fraction in terms of it before adding and simplifying

$$\frac{4}{15}+\frac{7}{30}=\frac{8}{30}+\frac{7}{30}=\frac{15}{30}=\frac{1}{2}$$

The 30 here is the LCM of 15 and 30. We could equally well use 15×30 = 450 but all the numbers get very large so it is nicer to use the smallest possible common multiple.

## Subtraction

Subtraction uses a similar method: $\dfrac{3}{5}-\dfrac{2}{5}=\dfrac{3-2}{5}=\dfrac{1}{5}$

$$\frac{2}{3}-\frac{1}{5}=\frac{10}{15}-\frac{3}{15}=\frac{10-3}{15}=\frac{7}{15}$$   (15 is the LCM of 3 and 5)

## Multiplication.

Very easy – "multiply the numerators (tops), multiply the denominators (bottoms)":

$$\frac{2}{3}\times\frac{5}{7}=\frac{2\times5}{3\times7}=\frac{10}{21}$$

but with mixed numbers, must convert to improper fractions first – see (b) above.

$$1\frac{1}{3}\times3\frac{1}{5}=\frac{4}{3}\times\frac{16}{5}=\frac{64}{15}=4\frac{4}{15}$$

To multiply by an integer, just multiply the numerator by it: $2\times\dfrac{2}{3}=\dfrac{4}{3}$

*You must know what this means:*

$\dfrac{1}{2}\times\dfrac{1}{4}=\dfrac{1}{8}$ means "if I have half a cake, and cut it in 4 equal parts, one of these parts is $\dfrac{1}{8}$ of a cake".

Remember, multiplying by $\dfrac{1}{4}$ is the same as dividing by 4.

## Reciprocals

Find the reciprocal of a fraction by swapping the numerator and denominator

The reciprocal of $\dfrac{2}{3}$ is $\dfrac{3}{2}$ (which means that $\dfrac{3}{2} = \dfrac{1}{\left(\dfrac{2}{3}\right)}$ )

The reciprocal of 2 is $\dfrac{1}{2}$ and vice-versa.

The reciprocal of $x$ is $\dfrac{1}{x}$ (use $x^{-1}$ button on calculator).

## Dividing fractions

To divide by a fraction, multiply by its reciprocal (so turn the second fraction upside down, then multiply)

$$\frac{3}{4} \div \frac{1}{8} = \frac{3}{4} \times \frac{8}{1} = \frac{24}{4} = 6 \quad (8 \text{ is the reciprocal of } \frac{1}{8})$$

*You must understand what this means:* "if I have ¾ cake and people want 1/8 cake each, I can feed 6 people" (the "filling containers" idea of division) or "How many eighths go into three quarters?"

This is similar to the calculation $12 \div 2 = 6$ which means "if I have 12 cakes and people want 2 each, I can feed 6 people"

To divide by an <u>integer</u>, multiply the denominator by it (or multiply the fraction by the number's reciprocal as above) eg

$$\frac{1}{2} \div 3 = \frac{1}{2 \times 3} = \frac{1}{6} \quad \text{or} \quad \frac{1}{2} \div 3 = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

# *Ratio and proportion*

Ratios enable us to divide collections of objects into groups or to describe the relative size of groups

- o The ratio 1:2 shows that out of a total of 3 "parts", I have 1 of one kind and 2 of another.

- o The ratio 1:2:4 shows that out of a total of 7 "parts", I have 1 of one kind and 2 of another and 4 of a third kind.

**Example**

30 people are split into 2 groups, A and B, in the ratio 10:20.

How many are in each group?

- ➢ A has 10, B has 20.

**Simplest form**

Just like equivalent fractions, we can multiply or divide all the numbers in a ratio <u>by a constant</u> without changing their relative proportions.

   e.g.  2:3 = 4:6 = 6:9 etc

To reduce to the simplest form, divide each side by any common factor.

   e.g.  24:20   has a common factor of 4  (it is 6×4 : 5×4), we divide each side by 4 to get 6:5

**Finding components when the total is given**
**Example**
20 people are split into 2 groups C and D in the ratio 9:1.  How many are in each group?

- ▪ Add the ratio numbers, 9+1 = 10 "parts".
- ▪ 20 people ÷ 10 parts = 2 people per part.
- ▪ Multiply the ratio by the size of a part, [9:1] * 2 people = 18:2.
    - o We have 18 people in group C and 2 in group D.

**Example**
A sponge cake recipe specifies flour, margarine and sugar in the ratio 4:2:1 by weight.

I want to make a 3.5kg cake.  How much of each ingredient do I need?

- • Add up the ratio numbers, 4+2+1 = 7 parts
- • Divide total mass by number of parts to get the mass per part,  3.5 ÷ 7  = 0.5 kg/part.
- • Multiply the ratio numbers by the mass per part, [4:2:1] ×0.5 gives
    - o 4×0.5 = 2 kg flour
    - o 2×0.5 = 1 kg margarine
    - o 1×0.5 = 0.5 kg sugar

**Finding the total when a component is given**
**Example**
Stainless steel contains iron, chromium and nickel in the ratio 74:18:8

   (a) If I have 90 kg chromium, how much stainless steel can I make?
- Add up the parts: $74+18+8 = 100$
- Total mass = $\dfrac{100}{18} \times 90 = \dfrac{9000}{18} = 500\,\text{kg}$

      (or think: mass of 1 part = $\dfrac{90}{18} = 5$ kg, therefore 100 parts = 500 kg).

   (b)  How much nickel do I need?
      Either:

      Calculate total mass = 500 kg.   $\dfrac{8}{100}$ of this is nickel, mass $\dfrac{8}{100} \times 500 = 8 \times 5 = 40\,\text{kg}$

      Or
      Each part = 5 kg,  8 parts = 40 kg.


**Direct proportion**
Two quantities are said to be in *direct proportion* when the ratio of one to the other is constant.

For instance, the power a solar cell can produce is proportional to its surface area.
A cell with 0.3 $m^2$ area produces 15 Watts.

   &#10148;  If I want to generate 300 Watts, I will need $\dfrac{300}{15} \times 0.3 = 6$ $m^2$

# *Formulae, equations and algebra – why to do things with letters*

## Formulae, using letters instead of numbers

Why use letters?

Suppose you are running the accounts for a company.  You know how much money has come into your company over the last year (income) and how much you have spent (expenditure) and you put the numbers into a spreadsheet, eg
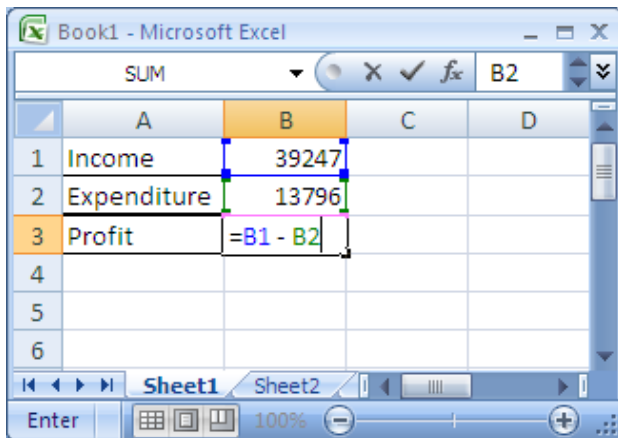


Here the "`= B1-B2`" is a formula that will calculate the profit as the number in cell B1 minus the number in cell B2.    When I finish entering the formula it automatically does the calculation:
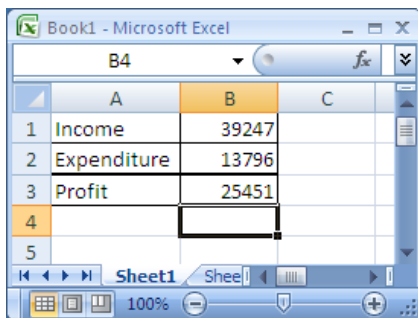


and if I change one of the numbers, Excel updates the answer by itself:
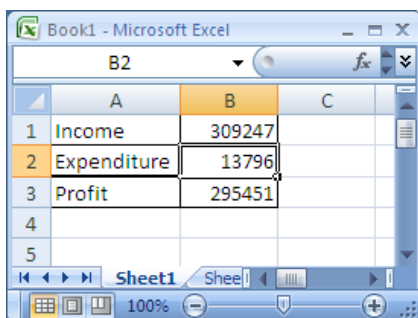


That's better!  Much more profit!

Mathematically, we could write this formula as `profit = income - expenditure`

People often want to do huge and complicated calculations and they get tired of writing "expenditure" so it is usual to pick just one letter.

We might write

$$P = I - E$$

We can see immediately that if we were told that $I = 25$ and $E = 10$ the profit would be

$P = 25-10 = 15$.

---

*A formula is just a mathematical description of how to do a calculation.*

*You put numbers instead of the letters on the right-hand side and get a value out for the left-hand letter.*

*We call the letters "parameters" or "variables" since they are not a fixed number – they can take any value.*



*It may help to think of the letter as a bucket.*
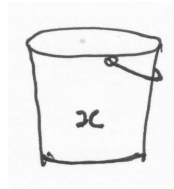
*It has a name painted on the side (x, y z etc) and there is a number carried inside it. Often we do not know the number and have to do calculations just using the bucket's name. Sometimes we can solve an equation – this tells us the number inside the bucket.*

---

## Re-arranging formulae

Often we have a formula and know the answer (left-hand variable) but not one of the input (right-hand) parameters. We then need to re-write the formula so the unknown parameter is on the left.

The procedure is the same as when solving equations (below).

**Example**

"Rearrange $P = I - E$ to make $I$ the subject."

We want $I$ by itself.

We spot that we can add $E$ to the right hand side to make it $I - E + E$ which is just $I$.

We have to do the same to the left-hand side (otherwise we have abused the = sign and the formula is rubbish) so we write

$$P + E = I - E + E = I$$

And then put the subject `I` on the left so its intention (Hey! I'm a formula for calculating `I` !) is obvious:

```
I = P + E
```

# *Units in formulae.*

When you do calculations using measurements that have units, the units add, multiply and divide just like the numbers. The formula above, `I = P + E`, has pounds plus more pounds giving an answer in pounds.

Similarly if I use the formula

$$S = \frac{D}{T}$$

to calculate speed from distance and time, my speed units depend on the distance and time units.

**Example:**

I run 105 metres in 15 seconds.

My speed is $\dfrac{105\text{m}}{15\text{s}} = 7$ m/s   (105÷15=7 and metres ÷seconds = metres per second, m/s)

> For **unit conversions** – see chapter 3.

# *Equations*

Equations look like formulae but their purpose in life is different.

> A formula contains several variable letters so if we know one parameter, we can easily calculate another, working from the right-hand side to the left-hand side.

> An equation contains just one kind of letter (often *x*) and if we are sufficiently cunning we can find a value for it such that the equals sign is valid (i.e. the value to its left and the value to its right are the same).

GCSE really only gives you an introduction to the simplest equations (in preparation for A-level work where you would learn about the more interesting equations that occur all the time in science and engineering). The mathematics of solving equations is known as *algebra.*

There are 3 ways of solving equations:
- Using algebra
- By plotting a graph and finding a coordinate where lines cross
- Trial and improvement

### *Rules of algebra*

An equation contains an equals sign "="

The values on each side of the "=" are, by definition, "equal" and we must do nothing that spoils this equality. Anything we do to the values to left of the "=", we <u>must also do to the values on the right</u>.

For instance, if we know $x = 3$ we could also say :

- $x + 1 = 3 + 1 = 4$ (adding 1 to each side)
- $x - 3 = 3 - 3 = 0$ (subtracting 3 from each side)
- $2x = 2 \times 3 = 6$ (multiplying each side by 2)
- $\dfrac{x}{4} = \dfrac{3}{4}$ (dividing each side by 4)
- $\dfrac{1}{x} = \dfrac{1}{3}$ (taking the reciprocal of each side)
- $x^2 = 3^2$ (squaring each side)
- $\sqrt{x} = \sqrt{3}$ (square rooting each side)

**Using algebra to solve an equation**

Use the above rules to reverse the stages that defined *x* in terms of the equation e.g.

To solve $2x + 1 = 5$ :

"subtract 1 from each side" $2x = 5 - 1 = 4$

"divide by 2" $x = \dfrac{4}{2} = 2$

nb. Do **not** write $2x + 1 = 5 - 1 = 4 \div 2 = 2 = x$. The string of "=" signs tells us that all the expressions have the same value but this is untrue (5-1 is not 2, for instance). Write it out on three separate lines.

*Do not try to do too much in your head –*

*write out the "working", this is what paper is for!*

*Show the working so you get some marks even if the final answer is wrong.*

## Graphical solution of equations

Sometimes problems can be solved approximately by drawing lines on a graph.  This is very useful if we are working with measurements (e.g. a table of speed versus age) that are not defined as a formula or equation.

Example.

An intrepid explorer finds himself between a polar bear and his rifle.   He is 200m from the bear and his rifle is 300m from the bear.  The bear can cover 200m in 15 seconds and the explorer can cover 100m in 25 seconds.  Does he get eaten?

We draw a graph of distance from the bear's starting point against time:
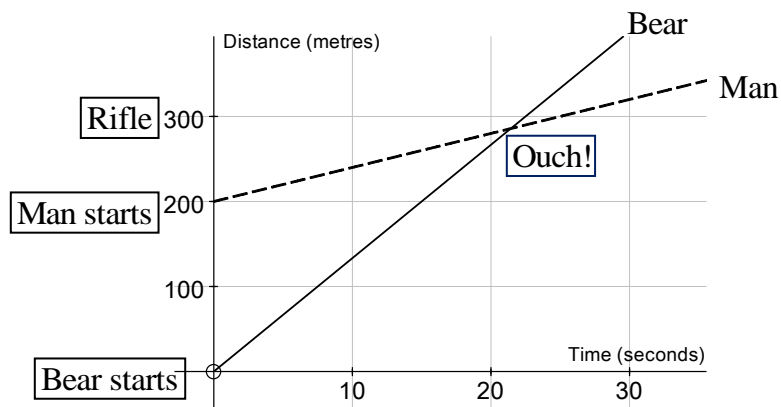


The lines cross after about 21½ seconds…he doesn't make it to his rifle in time.

# *Trial & improvement to solve equations*

We use a calculator to try several values and find which is closest to what we want.

**Example**

" $x^3 + x = 4$ has a solution between *x*=1 and *x*=2.   Find *x* to 1 decimal place accuracy."

Make a little table so our working is clear.

| x | $x^3 + x$ | |
|---|---|---|
| 1 | 2 | Too low |
| 2 | 9 | Too high |
| 1.5 | 4.875 | Still too high |
| 1.3 | 3.497 | Too low |
| 1.4 | 4.144 | Too high |
| 1.35 | 3.81 | Too low, so solution >1.35, rounds up to 1.4 |

- I have started with the two values given in the question.
- Next I will try one between them (1.5).
- That's too high, so try 1.3.
- Too low, try 1.4.

To one decimal place, the answer will be either 1.3 or 1.4 – but which?

1.4 looks closer from the table but to be sure, we need to know whether the answer is:

- above 1.35 (so will round up to 1.4 to 1 decimal place), or
- below 1.35 (so will round down to 1.3 to 1 decimal place)

So we try 1.35.  The answer is <4 so we know the solution is between 1.35 and 1.4.  It therefore rounds to **1.4**.

(All we are doing here is finding where two lines cross (as in the Polar Bear example) but we are avoiding actually having to draw the lines). If we drew it the graph would look like this:
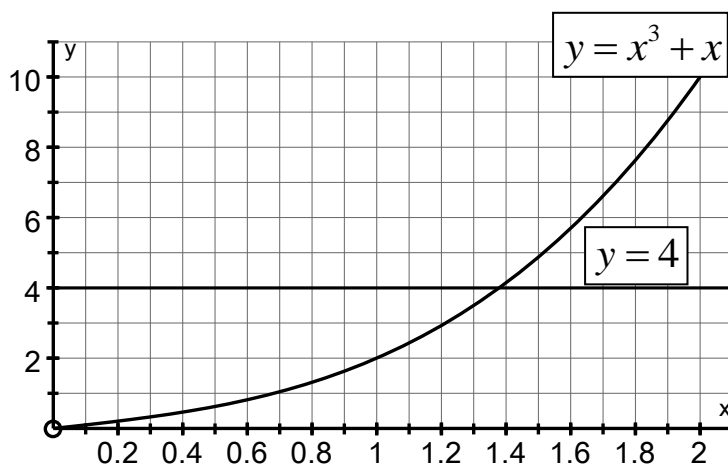
# *Further algebra*

The techniques used to solve more complicated equations include simplifying, multiplying out and factorising. At GCSE level these each are made into simple little questions – you never see a really complicated equation so may wonder about their purpose!

The rules for multiplying, dividing and simplifying fractions are the same in algebra (using letters x, y etc) as when just using numbers.

## Indices.

Rules:

1. $x^a \times x^b = x^{a+b}$    e.g. $10^3 \times 10^2 = 10^5$

2. $x^a \div x^b = x^{a-b}$    e.g. $10^3 \div 10^2 = 10^1$

3. $\left(x^a\right)^b = x^{ab}$    e.g. $\left(10^2\right)^3 = 10^6$

4. $\left(xy\right)^n = x^n y^n$    e.g. $20^3 = 2^3 \times 10^3 = 8 \times 1000 = 8000$

5. $x^{-a} = \dfrac{1}{x^a}$    e.g. $10^{-3} = \dfrac{1}{10^3}$

5. $x^0 = 1$    e.g. $10^0 = 1$

**Examples:**

(1) $3 \times 10^2 \times 5 \times 10^4 = 15 \times 10^{2+4} = 15 \times 10^6$. If the question asked for an answer in standard form you would need the "number part" to be between 1 and 10, so we write it as $1.5 \times 10^7$

(2) $2x^2 \times 3x^3 = 6x^5$

(3) $\dfrac{y^8}{y^3} = y^{8-3} = y^5$

(4) $4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

(5) $\dfrac{6x^3 y^5}{2x^2 y} = 3xy^4$

(6) $\left(3pq^3\right)^2 = 9p^2 q^6$

➢ When a question says "simplify" it often means you to do a multiplication or division as above

## Multiplying out brackets

Multiply all everything in the first bracket by each term in the second

Eg. $(10+2)(4+1) = 10 \times 4 + 2 \times 4 + 10 \times 1 + 2 \times 1$

$$= 40 + 8 + 10 + 2 = 60$$

(Remember *FOIL: First, Inner, Outer, Last*)

The process is the same when using letters. Examples:

$$2x(3x+5) = 6x^2 + 10x$$

$$(x+2)(x+3) = x^2 + 2x + 3x + 6 = x^2 + 5x + 6$$

### *Collecting terms*

You can only combine powers of x into a single term if they are the same power, e.g.:

➤ $2x^2 + 3x^2 = 5x^2$ (How many $x^2$ have I got? $2+3=5$)

➤ (whereas $x^2 + x^3$ cannot be simplified – we could "factorise" it as $x^2(1+x)$ - see below - but not make it into just one term).

This is known as "Collecting terms".

Example (note the "+3y"):

$$8(x+3y) - 3(2x-y)$$
$$= 8x + 24y - 6x + 3y$$
$$= 2x + 27y$$

## Factorising

➤ Factorising is the opposite of multiplying out.

Write down any factor that is common to each term (number and latter parts). Then write a pair of brackets and, in them, put the multiplier you need to convert your common term into the original expression.

**Examples.**

(1) $x^2 + x^3 = x^2(1+x)$ ($x^2$ is the common factor)

(2) $8x^3 + 3x^2 + 2x = x(8x^2 + 3x + 2)$ (x is the common factor)

(3) $2xy^2 + 6x^2y - 8xy = 2xy(y + 3x - 4)$ (2xy is the common factor)

# *Sequences*

A sequence is a list of numbers following a rule, for instance:

```
1   3   5   7   9        (odd numbers, increasing by 2 each time)
1   4   9   16           (square numbers)
2   4   8   16           (powers of 2, doubles each time)
1   10   100             (powers of 10)
1   3   6   10           (triangle numbers, like a triangular box of cannon balls – increases by
                          2, 3, 4 etc)
```

For any of these you need to be able to find the next term.
For sequences with constant interval, you must be able to find a formula.

**Example**
- The first 4 terms in a sequence are 32 29 26 23.
- Write down the next term
- Write the 10$^{th}$ term
- Find a formula for the sequence in terms of the position number *n*.

We see the terms decrease by 3 each time so the next term is 20.
We could find the 10$^{th}$ term by writing a long list
```
32 29 26 23 20 17 14 11 8 5
```
So the tenth term is 5.  Alternatively, find and use the formula.

To find the formula, either:
- note that the first term (n=1) has value $u_1$ = 32 and the step is -3 (going down each time) so
  our formula will be $u_n = -3n + \text{something}$

  To get the correct first term,  $32 = -3 \times 1 + \text{something}$,  32+3 = something = 35 and our
  formula is    $u_n = -3n + 35$

Or

- go back one step to find a "zeroth" term so the sequence is [35] 32 29 26 23 ….
  (so at n=0 the term would be 35).   The sequence is then obviously  $u_n = -3n + 35$

# Chapter 2: Graphs

## *Pie charts*

On a pie chart the area of each sector is proportional to the number of items it represents.   Usually a pie chart is circular and the sector angle is then proportional to number of items.

There are 2 types of question:

    (a)  Given a table, draw the chart

    (b)  Given a chart and one entry in a table, fill in the rest of the table

Type (a)   A vet sees in the course of one day 1 dog, 2 cats, 4 rabbits and 5 goldfish.  Draw a pie chart to show this.

First make a little table:

| Animal | Number | Angle |
|---|---|---|
| Dog | 1 | $\dfrac{1}{12} \times 360 = 30°$ |
| Cat | 2 | $\dfrac{2}{12} \times 360 = 60°$ |
| Rabbit | 4 | $\dfrac{4}{12} \times 360 = 120°$ |
| Goldfish | 5 | $\dfrac{5}{12} \times 360 = 150°$ |
| **Totals:** | **12** | $360°$ |

The total number of animals is 12 so the angle for each animal is $\dfrac{1}{12} \times 360°$
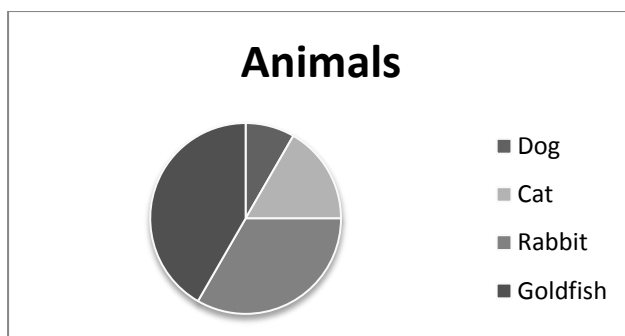
Type (b)  A vet shows the animals he has seen in the past week on a pie chart.

## Number of animals seen this week



■ Horses
■ Cows
■ Sheep
■ Llamas
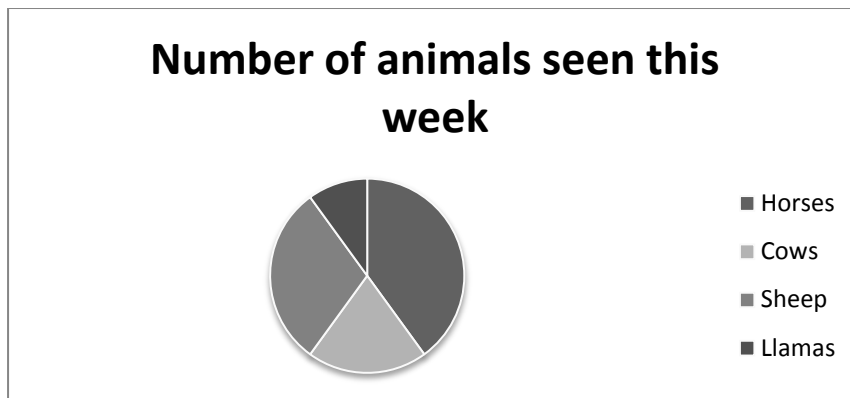
The angles for horses, cows, sheep and llamas are $144°$, $72°$, $108°$, $36°$.  Given that the $144°$ represents 12 horses, how many of each other animal has he seen?

$144° \div 12 = 12°$ per animal

| Cows | $72° \div 12° = 6$ cows |
|---|---|
| Sheep | $108° \div 12° = 9$ sheep |
| Llamas | $36° \div 12° = 3$ llamas |

# *x-y graphs*

## Coordinates

Cartesian coordinates are named after the French mathematician Descartes.   You can have two-dimensional coordinates (*x*,*y*) or three-dimensional (*x*, *y*, *z*).
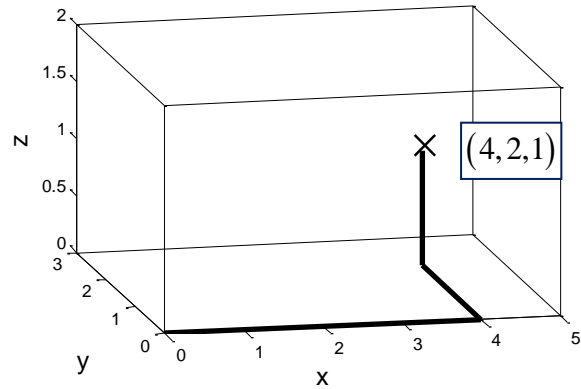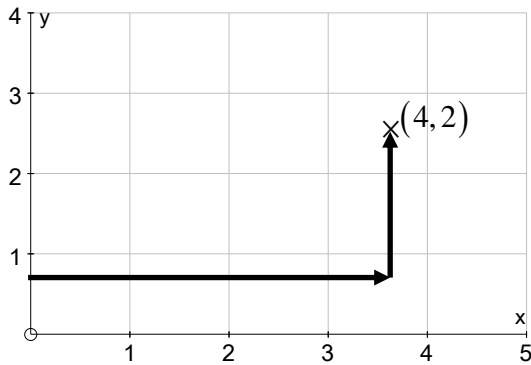
## Equations of lines

Straight lines are defined by an equation such as $y = 2x + 1$.

- Although we call this an "equation", we cannot solve it to find an *x* value because with two unknowns (*x* and *y*) we need two equations to get a solution!

- Instead, it acts as a formula.  If we know x we can find y.
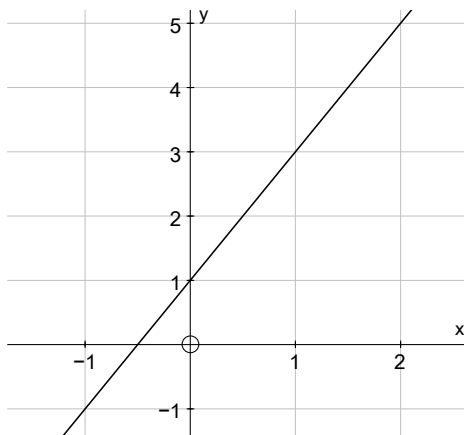
We see that the "1" is the y-coordinate at x = 0 and the 2 is the gradient of the line (if I move right 1, I go up 2).

## Drawing straight lines on graphs

Given the equation of a line, to draw the line we must find at least two points that it passes through. We can do this by choosing an *x*-coordinate for each point, then using the equation to find *y* there.

**Examples.** Plot the lines:

(a) $y = x$

(b) $y = 2x - 3$

(c) $y = -\dfrac{1}{2}x + 2$

(d) $x = 2$

(e) $y = -1$

**(a)** $y = x$. We could choose to have *x*=0 at our first point. Then since $y = x$, $y = 0$ and the point has coordinates (0,0).

We could choose to have *x*=2 at our second point. Then since $y = x$, $y = 2$ and the point has coordinates (2,2).

We can now draw a pair of axes, plot the two points and put a line through them.
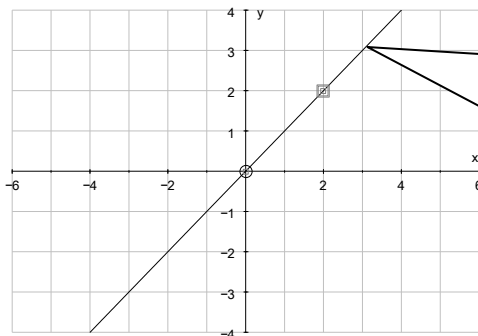


We can write this as $y = 1x + 0$

The "1" tells me it goes up 1 as it goes across 1 unit.

The "0" tells me it cuts the y-axis at y=0.

**(b)** $y = 2x - 3$. We could choose to have *x*=0 at our first point. Then since $y = 2x - 3$, $y = 2 \times 0 - 3 = -3$ and the point has coordinates (0,-3).

We could choose to have *x*=3 at our second point. Then since $y = 2x - 3$, $y = 2 \times 3 - 3 = 6 - 3 = 3$ and the point has coordinates (3,3).

We can now draw a pair of axes, plot the two points and put a line through them.
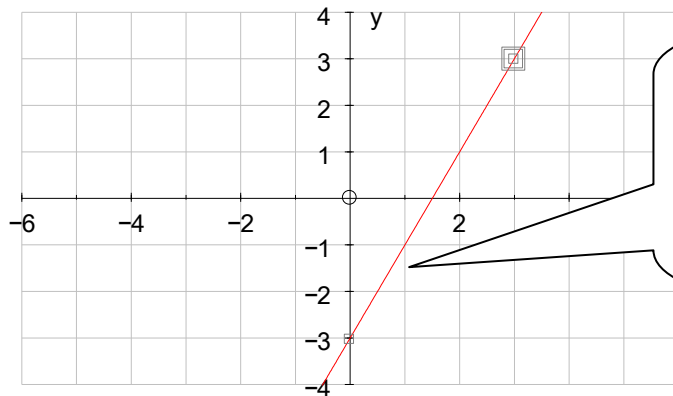


In the equation $y = 2x - 3$, the "2" tells me it goes up 2 as it goes across 1 unit.

The "-3" tells me it cuts the y-axis at y=-3.

**(c)** $y = -\dfrac{1}{2}x + 2$  We could choose to have *x*=0 at our first point.  Then since $y = -\dfrac{1}{2}x + 2$,

$y = -\dfrac{1}{2} \times 0 + 2 = 2$  and the point has coordinates (0,2).

We could choose to have *x*=4 at our second point.  Then since $y = -\dfrac{1}{2}x + 2$,

$y = -\dfrac{1}{2} \times 4 + 2 = -2 + 2 = 0$  and the point has coordinates (2, 0).
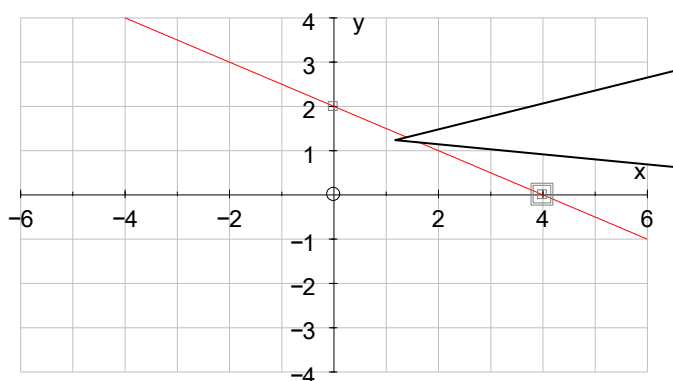


In the equation $y = -\dfrac{1}{2}x + 2$, the $-\dfrac{1}{2}$ tells me it goes up $-\dfrac{1}{2}$ (or down $\dfrac{1}{2}$) as it goes across 1 unit.

The "+2" tells me it cuts the y-axis at y=+2.

**(d)** $x = 2$ is a vertical line through any point where $x = 2$, eg through (2,0).

**(e)** $y = -1$ is a horizontal line through any point where y=-1, eg through (0, -1).

## Finding the equation of a straight line.

The general formula of a straight line is $y = mx + c$ where c is the *y*-value where it crosses the y-axis (i.e. at *x*=0) and m is the slope (how much higher it goes as one moves 1 unit to the right)
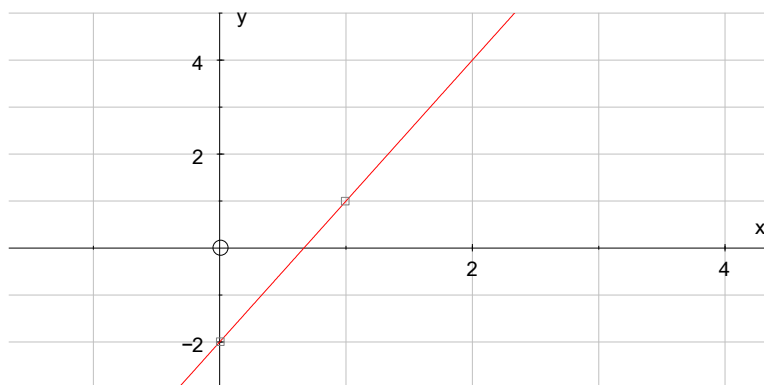


- For the line above it cuts the y-axis at y=-2, so we know *c* = **-2**.

- Moving across 1 to the right, the line goes from y=-2 to y=1, i.e. up 3 so the slope is $m = 3$.

- The equation is therefore $y = 3x - 2$

Having found the equation, try another x value in it to check it works.

e.g.   Try *x*=2.

Then $y = 3 \times 2 - 2 = 6 - 2 = 4$ and we can see that the point (2,4) is actually on the line, so we know the equation is correct.

## Plotting curved lines from an equation

Any formula that contains an $x^2$, $x^3$, $\sqrt{x}$ or $\dfrac{1}{x}$ will produce a curved line. You need to make a table of coordinates (see chapter 5, using a calculator), plot them and sketch a curved line through them.

The question will often then ask you to solve an equation by finding where lines cross.

For instance:
"Plot $y = x^2 - 4x + 3$, then solve $x^2 - 4x + 3 = 2$"
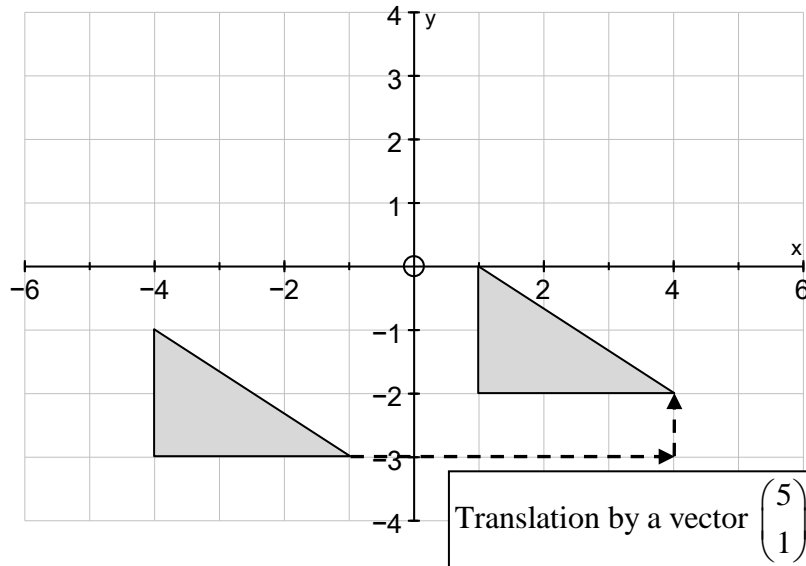
| x | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
|---|----|---|---|---|---|---|---|
| y | 8  | 3 | 0 | -1 | 0 | 3 | 8 |
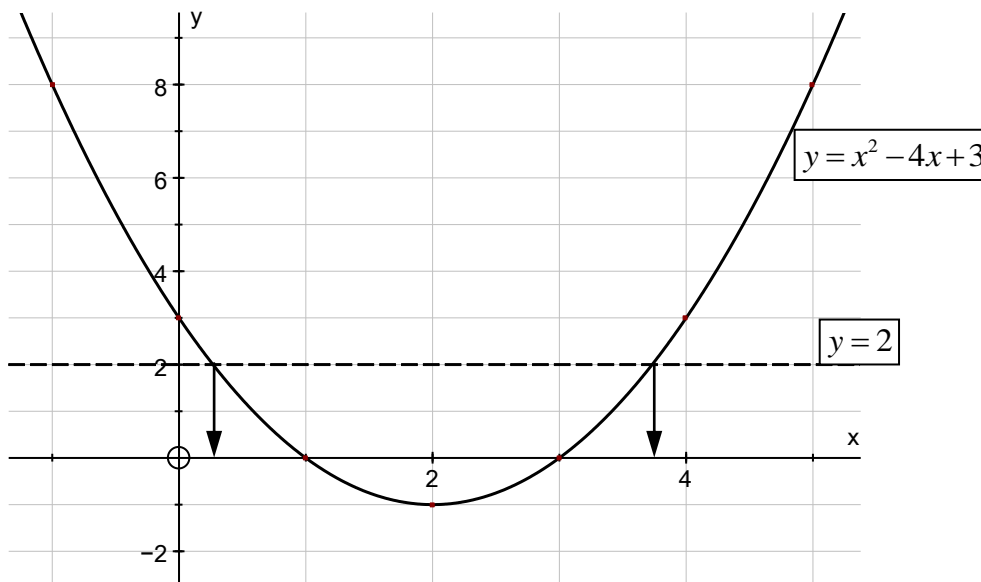


Looking at where the $y = x^2 - 4x + 3$ crosses the $y = 2$ line, I look down to the x-axis and read off
      x = 0.3  and
      x = 3.7  (roughly)

These are the two solutions for the equation.

# *Transformations*

## Translation

A translation shifts a shape



Translation by a vector $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$

## Enlargement

Enlargement has a centre and a scale factor.



original shape

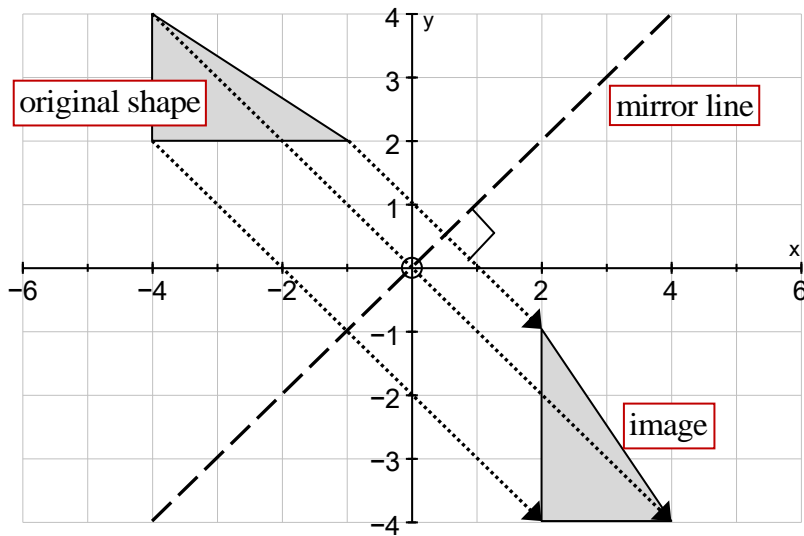new shape

Enlargement by factor of 2, centre (-5, -2)
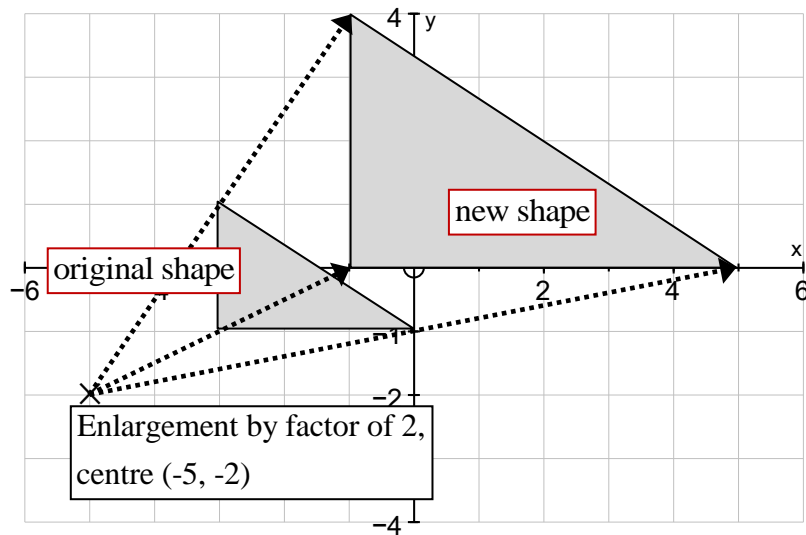
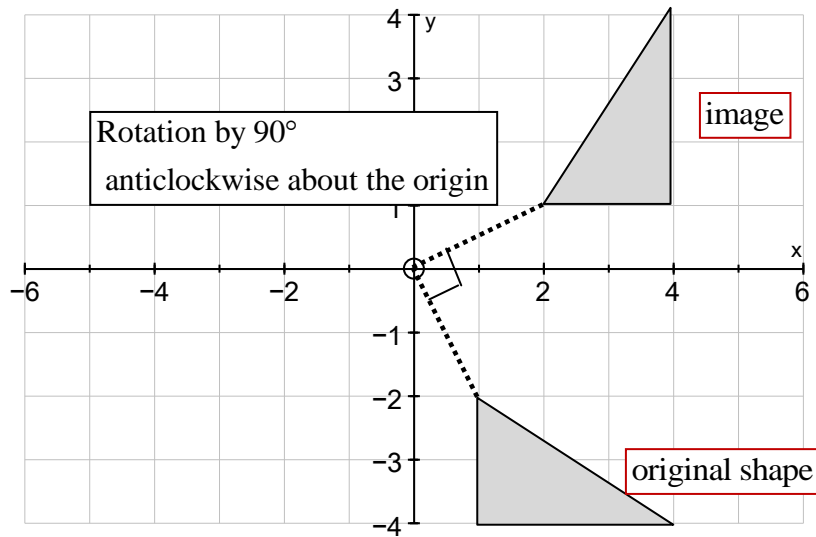# Reflection

Reflection makes a "mirror image" of a shape on the other side of a mirror line.
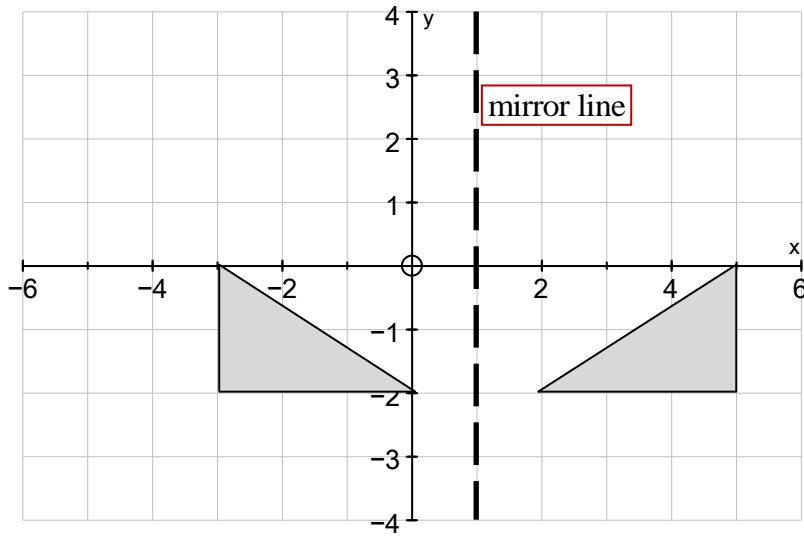


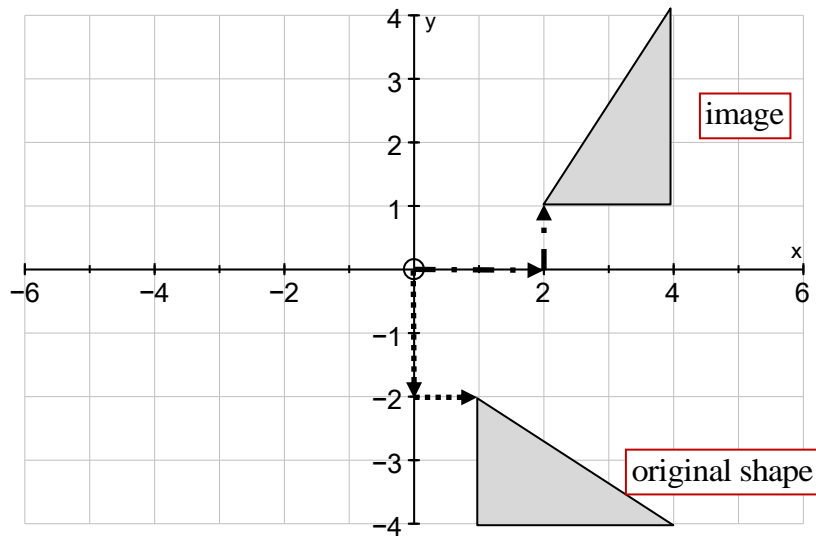If the line is sloping, you will need to draw the image carefully with lines crossing the mirror line at right angles:

# Rotation

Rotation uses a centre and an angle:



It is often convenient to think of the shape as being "held" by a dog-leg from the rotation centre. Rotate these legs to get the new coordinates of each point in your shape, then draw lines between the points:

# Chapter 3:  Drawing, geometry, shapes, area & volume, units, mass,

## *Drawing and measuring*

### Measuring the angle between two lines.

- The lines must cross and be long enough to reach the scale on the protractor.  If they are not, use a ruler and a sharp pencil (or thin pen) to extend them.

- The protractor can measure an angle clockwise from the left side or anti-clockwise from the right.  To use the right scale, make sure you "count up" the angle on the protractor scale that goes 0, 10, 20 etc (so a very small angle would be close to 0°).   Think whether you want an angle <90°, between 90 and 180°, or >180° and check you have a suitable answer.

- If you are asked for an angle >180°, measure the angle between the lines and subtract from 360°.

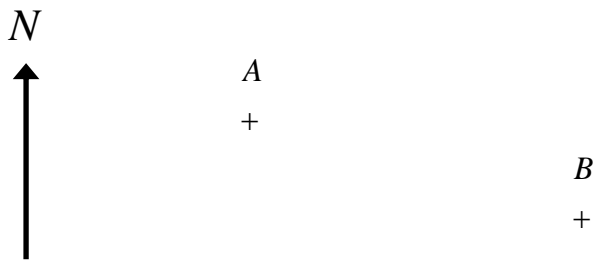### Bearings

All bearings are given as three digits e.g.  015°

Bearings are measured clockwise around from North so a bearing of 000° is due North and 090° is due East.
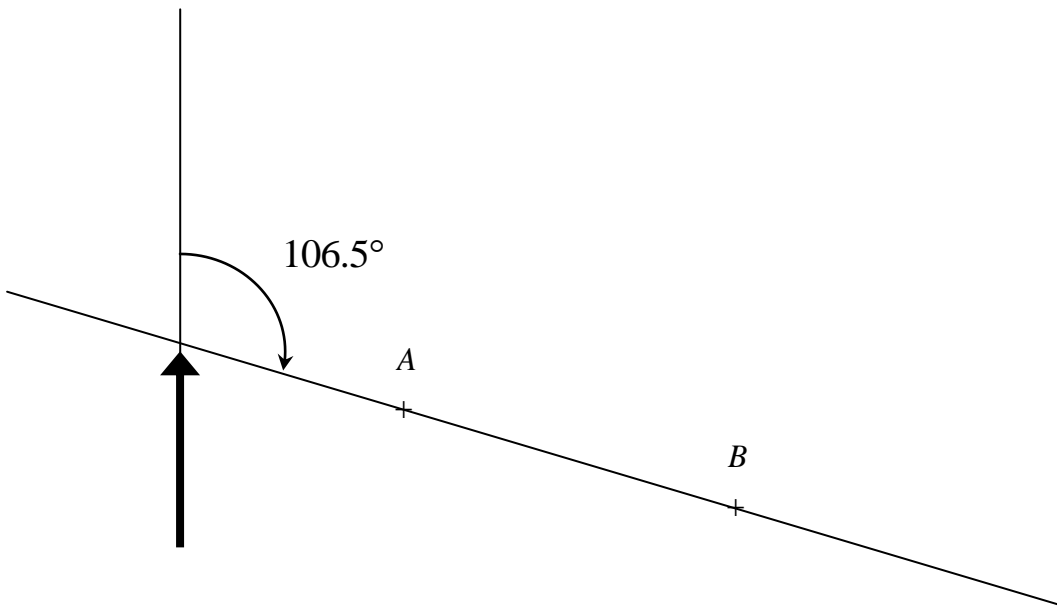


**Compass Rose**
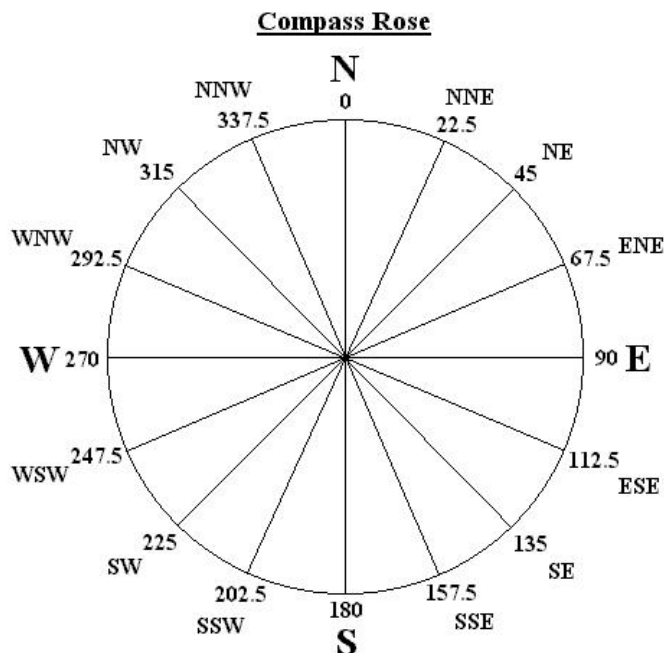
(image from http://www.ke4nyv.com/navigation.htm)

Example 1

"Measure the bearing from A to B"

$N$

A
+

B
+

You need to extend the North arrow and draw a line through A and B so it crosses the North line. Then measure the angle.  The angle is obviously more than 90°. If your angle is less, you have used the wrong protractor scale.

106.5°

A

B

Example 2.  If the question asked for the angle from B to A, imagine you are standing at B and facing A. You need the angle clockwise from North.   Measure angle x as above, then add 180° (or take 180, if need be, so your answer is not bigger than 360°)

$x$

A

B

$106.5° + 180° = 286.5°$

52
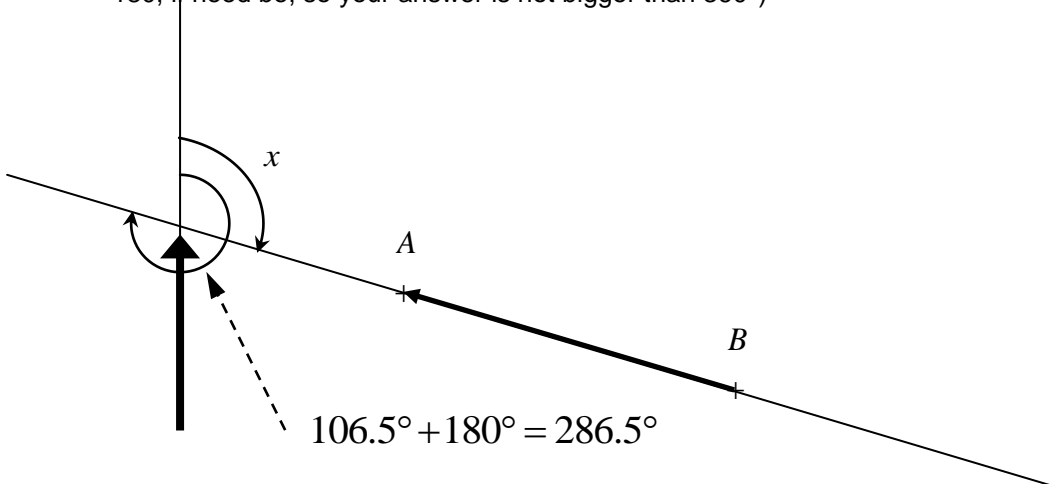
## Drawing a triangle, given 2 sides and an angle.

Usually one side is already drawn on the answer sheet (if not, draw it).  Your length must be accurate to the nearest millimetre.
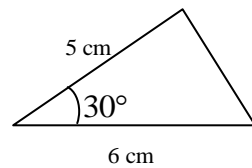
Use the protractor to measure the angle. Make a little mark, then (using the ruler) draw a long faint line through it and the starting point.

Set your compasses to draw a circle with the length of the third side.  Draw an arc that cuts your line.
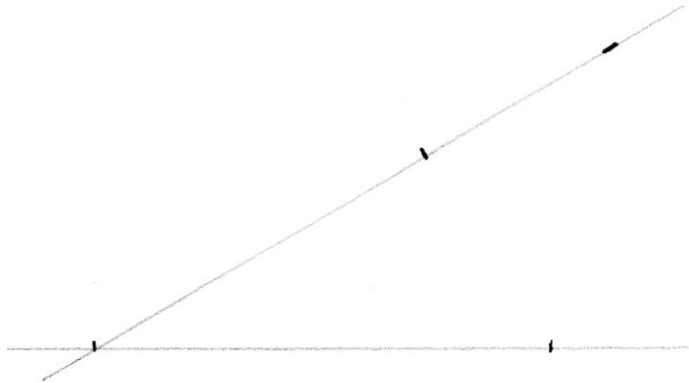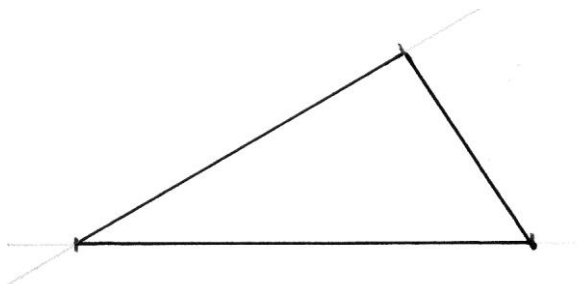
Put a line between the corner points.

Example 3

Draw the following triangle accurately



5 cm

30°

6 cm

Draw 2 lines at 30° to each other.   Measure and mark a point 6 cm along one, 5 cm along the other.
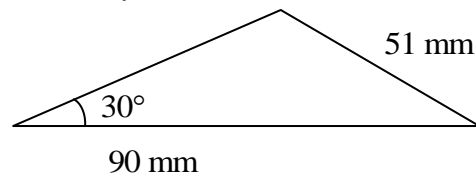


Then draw lines between the points.

Example 4
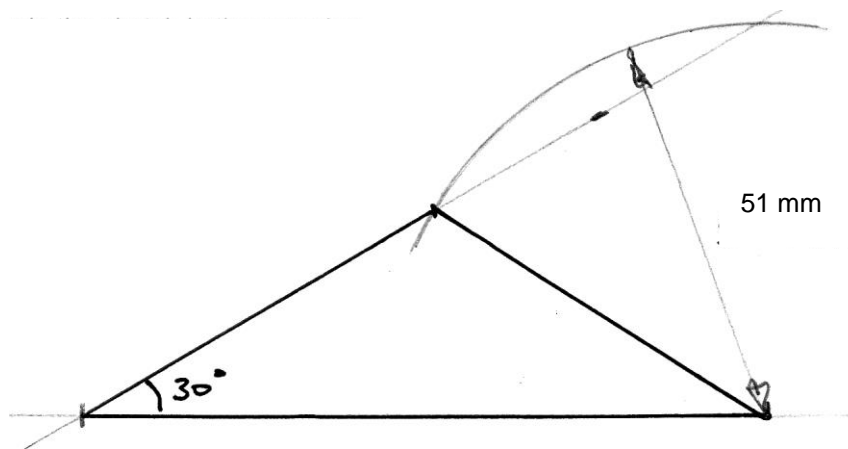
Draw the following triangle accurately

51 mm

30°

90 mm

- As before, start by drawing two lines at 30°.
- Measure a 6 cm bottom edge and mark its ends.
- Set the compasses to draw a circle of 4 cm radius and draw a circular arc centered on point B
- This will cross line AC at two points.  Draw a line through whichever looks most like the example in the sketch in the question.
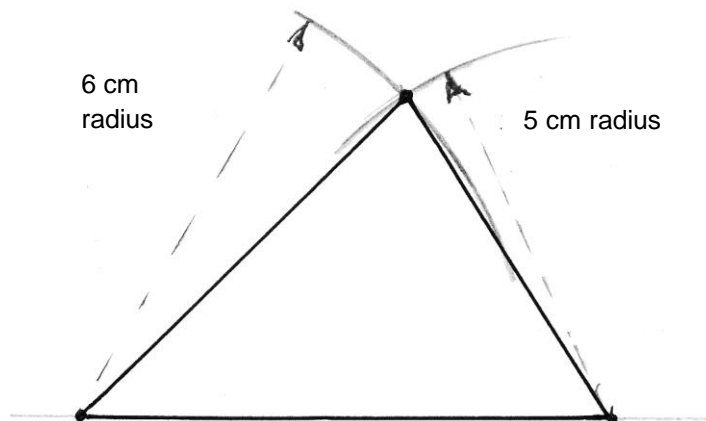
51 mm

30°

## Drawing a triangle given the lengths of 3 sides.

- Draw one side
- Use compasses to draw an arc with exactly the right radius for each other side.
- Where the arcs cross is the third corner.

Example 5

Draw a triangle with sides of length 7, 6 and 5 cm.

6 cm
radius

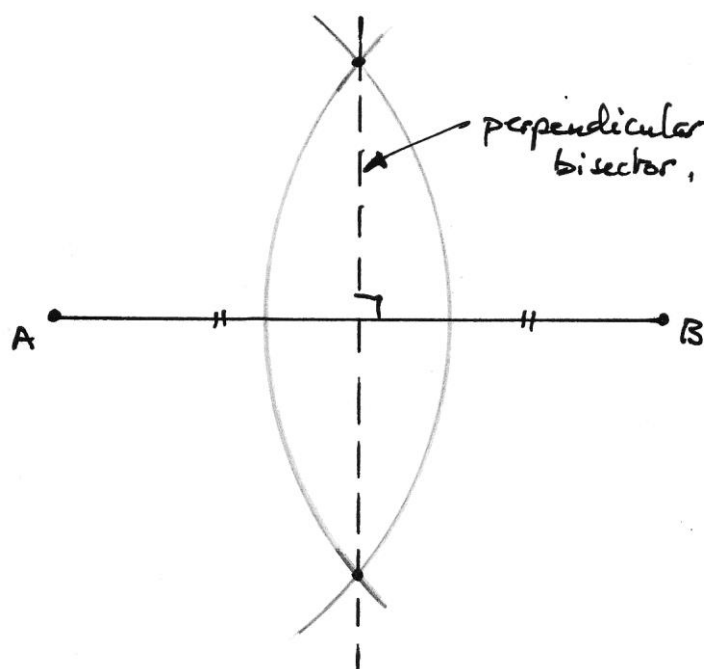5 cm radius

## Bisecting an angle

- Using the compasses, draw an arc centered on the point where the lines cross.
- Your arc crosses each line once. Put the compass spike on one crossing point and draw an arc (can be same radius as first, but not essential).
- Now very carefully without changing the compass width, put the spike on the other crossing point and draw another arc (must be same radius as the one before).
- Draw a straight line through the intersections.



same radii.
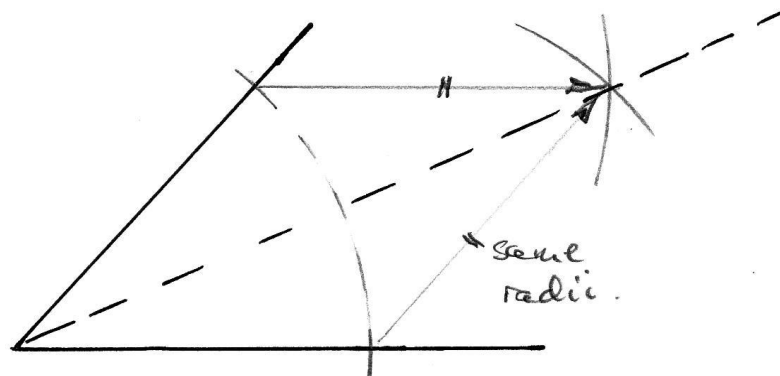
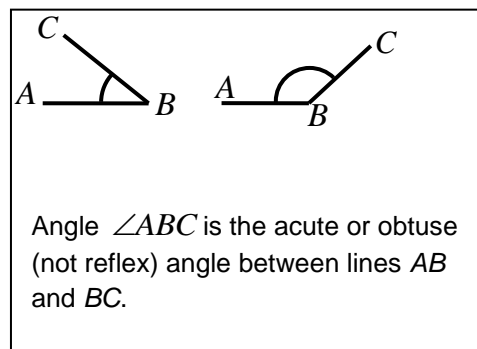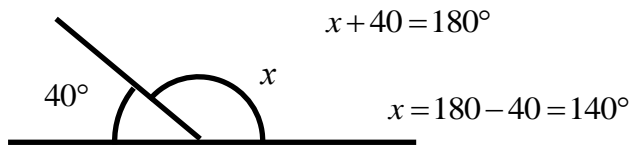## Drawing the perpendicular bisector of a line segment

- A line segment is a line between two points.
- A perpendicular bisector goes through the mid-point of this line and cuts it at right angles.
- You are expected to construct it using compasses. The examiners will look for your construction lines.
- Both arcs must be exactly the same radius.

This gives you the "locus of points equidistant from A and B". (A locus is a line defined by a series of points).
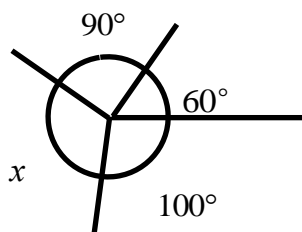


perpendicular bisector,

A          B

# *Angles*

1. Angles on a straight line add to 180°.

$$x + 40 = 180°$$

$$x = 180 - 40 = 140°$$

40°

$x$

Angle $\angle ABC$ is the acute or obtuse (not reflex) angle between lines $AB$ and $BC$.

$C$     $C$
$A$   $B$   $A$   $B$

2. Angles around a point add to 360°. Whenever possible show your working by writing an equation and solving it.
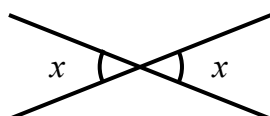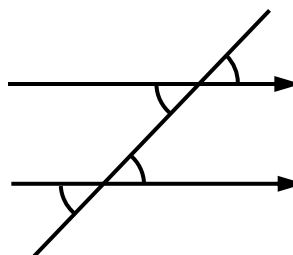
90°

60°

$x$

100°

$$x + 100 + 60 + 90 = 360°$$

$$x + 250 = 360°$$

$$x = 360 - 250 = 110°$$

3. When straight lines cross, the "vertically opposite" angles are equal:

$x$    $x$

4. When a straight line cuts a pair of parallel lines:
   (a) all the acute angles are equal
   (b) all the obtuse angles are equal
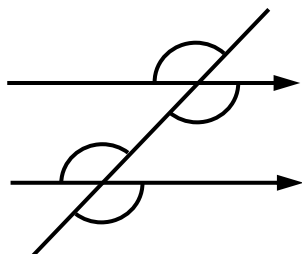
*Names for pairs of angles:*

Corresponding (F) angles are equal

Alternate (Z) angles are equal
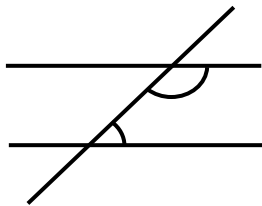
Supplementary angles add to 180°

**5. Angles inside a triangle add to 180°**
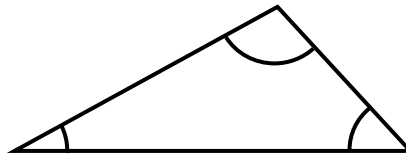
6. The exterior angle at one corner = the sum of the opposite two interior angles:

$100°$

$30°$

$100° + 30° = 130°$
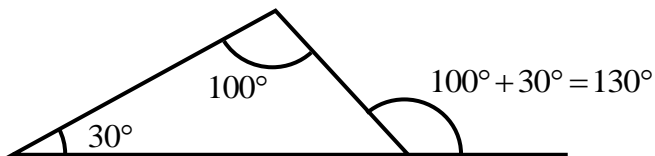
# *Names of shapes*

**Two-dimensional shapes**

Two-dimensional shapes often have names ending in "-gon".  The name identifies the number of sides.
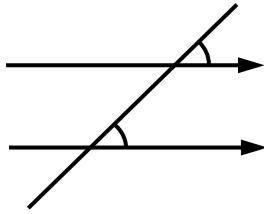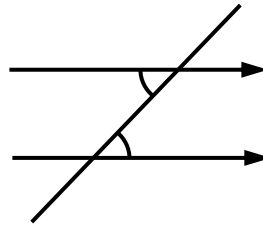
<u>One side:</u>

Circle

Ellipse
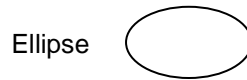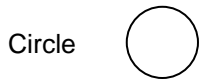
<u>3 sides (triangle):</u>

- Equilateral = all sides same length, all angles 60°.

- Isosceles = two sides same length, two angles equal

- Scalene = 3 sides all different lengths
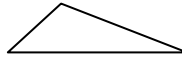
- Right-angle = one angle is 90°
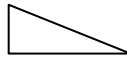
<u>Types of quadrilateral (4 sides):</u>
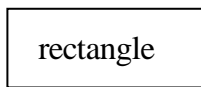
rectangle

square

rhombus
(a sheared square)
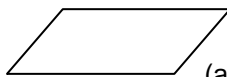
parallelogram
(a sheared rectangle)
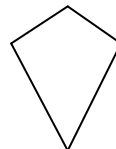
kite
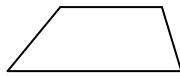
trapezium
(two parallel sides)

quadrilateral
(anything else)
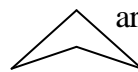
arrowhead

<u>Shapes with more sides</u>
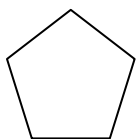
Pentagon
(5 sides)
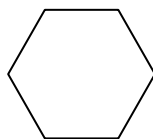
Hexagon
(6 sides)
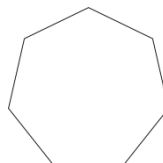
Heptagon
(7 sides)
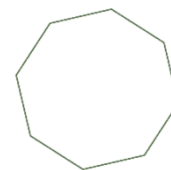
Octagon
(8 sides)

**Three-dimensional shapes**

- ➢ Cube  (all 6 sides are squares)
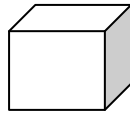
- ➢ Cuboid (a squashed or stretched cube, rectangular sides).

- ➢ Prisms (triangular, hexagonal etc): a two-dimensional shape extruded to make something 3-dimensional.
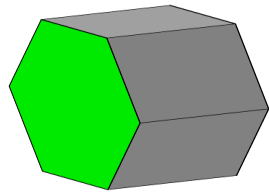
"The volume of a prism = (cross-section area)×length"

- ➢ Pyramids taper to a point
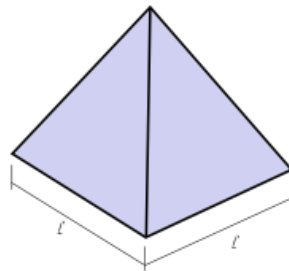
A square-based pyramid

Other three-dimensional shapes have names ending in "'-hedron".  The name identifies the number of faces. *Just for interest*:

- ➢ Tetrahedron

   (4 faces, each an equilateral triangle, 3 faces meet at each vertex)
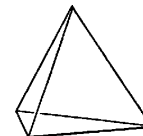
- ➢ Octahedron
   (8 faces, each an equilateral triangle, 4 faces meet at each vertex)
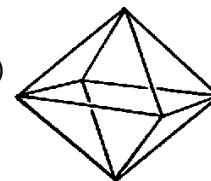
- ➢ Dodecahedron
   (12 faces, each a pentagon)
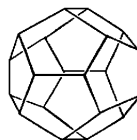
- ➢ Icosahedron
   (20 faces, each an equilateral triangle, 5 faces meet at each vertex)

(images from http://www.math.wichita.edu/history/Topics/geometry.html).
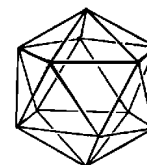
# *Area*

Area is "how much surface" something has.  Surfaces need painting (how much paint?) or carpeting (how much carpet?)



| Rolls of carpet. |
|---|
| Area = width×length |

We tend to distinguish between "the area" of a 2-dimensional object (eg a rectangle - has area but no volume) and "the surface area" of a 3-dimensional object (a dormouse).

A large surface area increases heat loss (eg the ears on a elephant).  A small surface area minimises heat loss (why dormice curl up and birds have thin legs).

|  |  |  |
|---|---|---|
| a dormouse curls up into a ball to reduce its surface area and stay warm | An elephant has large ears to increase its surface area and help it keep cool. | A satellite has solar panels with a large area to convert sunlight to electricity |

# Calculating the area of a shape.

(1) if "surface area", add up the areas of all the faces.

(2) for 2-dimensional objects:

- triangles:  area = $\frac{1}{2} \times \text{width} \times \text{height}$    (where "width" and "height" are measured at right

  angles to each other).
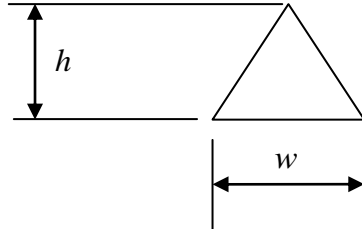


- All squares, rectangles and parallelograms,  $\text{area} = w \times h$ where $w$ = base width and $h$ = height in a direction at 90° to the base line
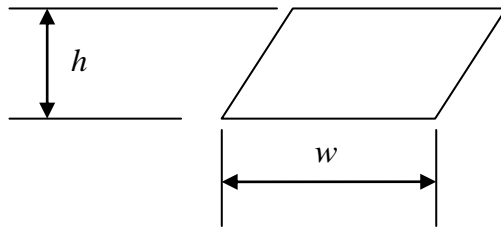


- trapezium:  area = $\text{width} \times (\text{mean height}) = \frac{w(a+b)}{2}$  (formula at start of exam paper)
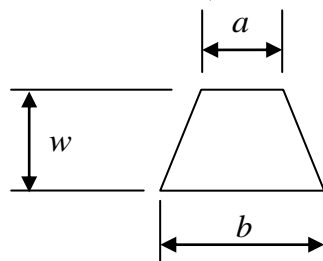


- circles:  area = $\pi r^2$   ( r = radius = half the diameter $D$ ).
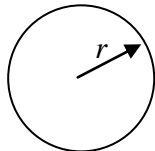


- Complex shapes:  divide the shape into rectangles, triangles and parts of circles, then add up the area of each (or subtract if a shape has been cut out and thrown away).

**Examples**

**(1)** A room is carpeted with Wilton carpet costing £30/m$^2$. The floor is 8 m long by 5 m wide. How much did the carpet cost?

Area = $8 \times 5 = 40 \, \text{m}^2$.

Cost = $40 \, \text{m}^2 \times £30/\text{m}^2 = £1200$

(2) A rectangular courtyard 5 m by 8 m is covered with gravel apart from a central 3 m diameter flower bed. What is the gravelled area?
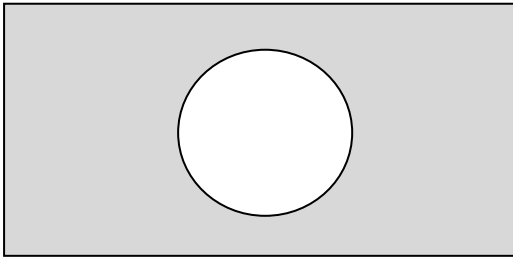


Rectangle: area = 5×8 = 40 m$^2$.

Circle, radius = half the diameter = 1.5 m

Gravelled area = $5 \times 8 - \pi \times 1.5^2 = 32.9 \, \text{m}^2$ (32.93 would also be a suitable answer, but unnecessarily precise given that gardens are not built to 4 significant figure accuracy).

**(3)** A Christmas present comes in a cubic box with sides 0.3 m long. What is its surface area?

Each side has area $0.3 \times 0.3 = 0.09 \, \text{m}^2$

All cubes have 6 sides, total area = $6 \times 0.09 = 0.54 \, \text{m}^2$

# *Calculating volume.*

Volume tells us "how much space something takes up". We can also use it as a measure of an amount of liquid or the capacity of a container.

> ➢ We measure volume in litres, cubic centimetres ($cm^3$) or cubic metres ($m^3$).

> ➢ 1000 $cm^3$ = 1 litre, 1000 litres = 1 $m^3$.

> Aside: When a car is described as having a 2 litre engine, this means that in one engine cycle (two rotations) the air in the cylinders is squashed to a volume 2 litres less than before, then expanded back again.



> A tall milk bottle holds 1 litre of milk.

Rules for finding volumes:

> ▪ For any kind of "prism" with a constant cross-section area, Volume = $A \times h$ where A is the end area and h is the height (or length) perpendicular to it. (Perpendicular means "at right angles"). See formula at start of exam papers.

> ▪ For more complex shapes, add up the volume of each simple part. You may get a shape made from cubic blocks and just need to count or calculate the number of blocks.

**Examples**

**(1)** An Olympic swimming pool is 50 m long, 25 m wide and 2 m deep. What is its volume?

Volume = $50 \times 25 \times 2 = 50 \times 50 = 2500 \text{ m}^3$

(Note units - area is length×length, units $m^2$ but volume is length×length×length, units $m^3$ ).

**(2)** A jug is 10 cm diameter (internally) and is filled with water to a depth of 12 cm. What volume of water is this?

Radius $r = \dfrac{\text{diameter}}{2} = 5 \text{ cm}$

Cross-sectional area = $\pi r^2 = \pi \times 25 \text{ cm}^2$

Volume = $25\pi \times 12 = 942 \text{ cm}^3$ to 3 significant figures

# Pythagoras' theorem

The lengths of the sides of a right-angled triangle are related by Pythagoras's theorem:

$$a^2 + b^2 = c^2$$

Where c is the length of the longest side.
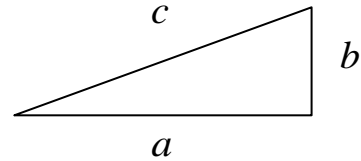
To find one of the shorter sides, take $b^2$ from each side:

$$a^2 = c^2 - b^2$$

**Examples**

   **(1)**  A builder is building a roof.  The roof is 6 m wide and 1.4m high at the apex.  How long should the rafters be?
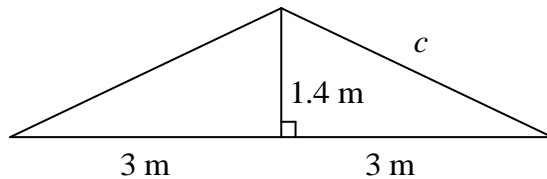
$$c^2 = 3^2 + 1.4^2 = 10.96$$

$c = \sqrt{10.96} = 3.3106$ m  (do [√] ANS = on calculator)

   **(2)** A joiner is making a door frame for a door.  The door is 500mm wide and 40mm thick.  The aperture needs to be slightly wider than this so the door does not jam when opening.  How wide does it need to be?
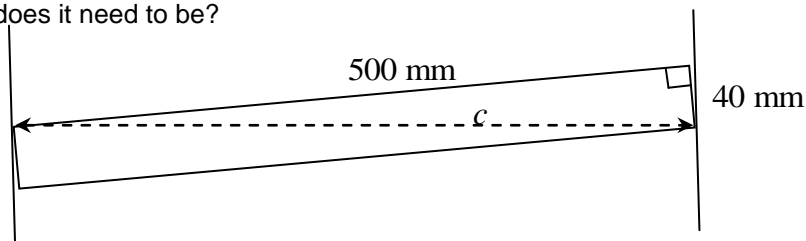
$$c^2 = 500^2 + 40^2 = 251600$$
$c = \sqrt{251600} = 501.6$ mm, he needs to make the frame at least 1.6mm wider than the door.

**(3)** I buy a sheet of plasterboard 1.2 m wide and want to get it home in my car. The car is only 1 m wide so it will have to sit at an angle. How high will one side be?
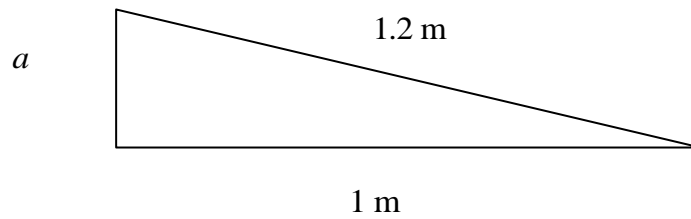


$$a^2 = 1.2^2 - 1^2 = 0.44$$

$$a = \sqrt{0.44} = 0.663 \text{ m}$$

Aside: there are many right angled triangles that have sides with integer (whole number) lengths. The simplest of these has side in the ratio 3:4:5 (so 3m, 4m and 5m or 6m, 8m and 10m).

These sets of three numbers that satisfy $a^2 + b^2 = c^2$ are known as Pythagorean triplets.

Other common triplets are (5, 12, 13), (7, 24, 25) and (8, 15, 17).

# *Units*

We can distinguish between

> ➢ Units for different types of thing (a volume unit is different to a speed unit, if you ask a barman for 5 miles/hour of beer you will get a funny look)

> ➢ Various units that all basically measure the same thing (metres, inches, feet, yards, fathoms, miles are all length measurements and we can easily convert one into another).

> ➢ Sub-divisions of units (millimetres and centimetres are just little bits of a metre)

## Imperial and metric units

For thousands of years, people used units that were pretty roughly defined.  An inch for instance (supposedly the length of the tip of one's thumb) could vary from country to country and be anywhere between 19 and 29 mm.

These old units have come to be known as "imperial" units.

In 1799 the French adopted a metric system that used multiples of the metre for all measurements instead of following the imperial practice of having large numbers of different length units.   In 1960 this metric system was defined in terms of international standards and adopted worldwide.  There are 7 basic SI units:

- Metre (length)
- Kilogram (mass)
- Second (time)
- Ampere (electric current, eg 13 amp fuse)
- Kelvin  (temperature, eg absolute zero is 0 Kelvin)
- Candela (brightness of a light, roughly one candle!)
- Mole (a counting number like a couple or dozen but much larger, $6.02 \times 10^{23}$ )

All the other metric units are derived from these (Length ÷ time gives speed in metres/second, for instance).

## Multiples of metric units

All metric units can take <u>prefixes</u> that make them into bigger or smaller units.  There are lots of these!

The list goes (and there are more at each end!):

| Prefix | Multiplies by | example |
|--------|---------------|---------|
| Giga | $\times 10^9$ | 1 giga-watt-hour (a lot of energy) |
| Mega | $\times 10^6$ | 1 megawatt (the power output of a large lorry engine) |
| **Kilo** | $\times 1000$ | 1 kilogram (1000 grams), 1 kilometre (1000 metres) |

| Deci | $\times 0.1$ | 1 dm = 10 cm, 1 decimetre³ (same as a litre) |
|---|---|---|
| **Centi** | $\times 0.01$ | 1 centimetre = $\dfrac{1}{100}$ m |
| **Milli** | $\times 0.001$ | 1 millimetre = $\dfrac{1}{1000}$ m, <br><br> 1 milligram = $\dfrac{1}{1000}$ g, |
| Micro | $\times 10^{-6}$ | 1 micrometer = 1 millionth of a metre |
| Nano | $\times 10^{-9}$ | 1 nanometre, nanogram etc |
| Pico | $\times 10^{-12}$ | 1 picofarad, picosecond etc |

You only need to know kilo-, centi- and milli- but the others will help you make sense of things you read about or hear.

| SI to imperial conversion summary | |
|---|---|
| Length | **25.4 mm = 1" (inch) exactly** <br><br> **1 metre = 39.3 inch** <br><br> **1.6 km = 1 mile   (or 8 km = 5 miles)** |
| Mass | **1 kg = 2.2 lb (pounds)** <br><br> **28 g = 1 oz (ounce)** |
| Volume | **1 litre = 1.76 pints** <br><br> **4.545 litres = 1 gallon** |

| Common multiples of imperial units | |
|---|---|
| **1 foot = 12" (inches),** | **1 yard = 3 feet** |
| **1 pound = 16 ounces (oz),** | **1 stone = 14 pounds (lb)** |

## Units of length

There are lots of units for length

**Imperial** (you only need to know the **bold** ones):

1 inch is now defined as being exactly 25.4 mm.

**12" (inches)  = 1 foot**

**3 feet = 1 yard**

6 feet = 1 fathom

440 yards = 1 furlong (quarter of a mile)

**1760 yards = 1 mile** (=5280 feet)

6080 feet = 1 nautical mile    (so 1 knot = 1 nautical mile per hour)

(and very historically, 1 cubit = 0.5 yard, 1 chain = 22 yards = 4 rods).

Also in astronomy, a light year is the distance light travels in a year = about 6 million million miles and a parsec is 3.26 light-years.

**Metric**

10 mm = 1 cm

1000 mm = 100 cm = 1m

1000 m = 1 km

**Conversions**

Nb.  You don't have to remember exact values, just approximate conversions

1 inch = 25.4mm = 2.54 cm

1 foot = 30.48 cm = 0.3048 m  and   1 metre = 3.2808 feet = 39.3 inches

1 mile = 1.6 km  (so 5 miles = 8 kilometres)

## Units of mass

(commonly called "weight" in GCSE questions)

### Imperial units

16 ounces (oz) = 1 pound (lb)

14 pounds = 1 stone

112 pounds = 8 stone = 1 hundredweight (cwt)

2240 pounds = 20 hundredweight = 1 ton

### Metric

1000 kg = 1 tonne

### Conversion, metric mass to imperial mass:

1 kilogram = 2.2 pounds  (1 kg = 2.2 lb),  1 pound = 453 grams = 0.453 kg

## Units of volume

20 fluid ounces (the volume of 20 ounces of water) = 1 pint

(nb an American pint = 16 fluid ounces so American pints and gallons are smaller than UK ones).

2 pints = 1 quart

8 pints = 1 gallon

1 "barrel" of oil = 42 US gallons = about 35 British gallons.  Nowadays the barrels one sees are usually larger, a typical metal barrel being 44 UK gallons or 55 US gallons:



### Memory aids

- A short ruler is 6", about 150 mm
- A long ruler is 1 foot, 12 ", about 300 mm

- If you are 6 foot tall, that is 1.83 m
- A 4 m wide roll of carpet will fit a room 13 feet wide.

- Runners who manage the "4 minute mile" are doing 1600 m in 4 minutes so 400 m/minute, equivalent to 100 m in 15 seconds.
- A good cyclist can do 25 miles in one hour (25 mph) which is 11.1 metres/second.
- A Boeing 747 flies at 565 miles per hour (910 km/h).  It is 70 m (231 feet) long, 64 m (211 feet) wingspan and weighs 180 tonnes (empty).

- A mouse weighs between 10 and 25 grams (0.3 to 0.9 ounce)
- 4 apples weigh about 1 pound
- A small bag of sugar weighs about 1 pound (453 grams)
- An 11 stone person weighs 154 lbs which is 70 kg.
- Most cars weigh between 800 and 1400 kg (0.8 to 1.4 tonnes)
- An African elephant can weigh up to 5400 kg.
- A 53 seater coach weighs about 11 tonnes.

- Plastic milk bottles hold  0.5 (short bottle), 1 (common tall) or 2 litres (wide tall).
- "A pint of water is a pound and a quarter,
        A litre is a pint and three quarters"
        ( meaning 1 pint = 20 fluid ounces, 1 litre = 1.76 pints ).

# *Unitary form*

Sometimes we are asked to choose which item to purchase – which is the best value?

We need to calculate the price <u>per item</u>, price <u>per unit mass</u> or price <u>per unit volume</u>.

This is known as converting the prices to unitary form.

**Example**

One can buy baked beans in 560 gram tins (42p), 750 gram tins (60p) and 2 kg tins (£1.50).  Which is the best value?

| Size | 0.56 kg | 0.75 kg | 2 kg |
|---|---|---|---|
| Cost | £0.42 | £0.6 | £1.4 |
| Cost £/kg | $\dfrac{£0.42}{0.56} = £0.75 \text{ /kg}$ | $\dfrac{£0.6}{0.75} = £0.8 \text{ /kg}$ | $\dfrac{£1.4}{2\text{kg}} = £0.7 \text{ /kg}$ |

# *Conversion between units*

## Unit conversion factors (units of same type but different names)

To convert a measurement from one system of units to another, we can use two methods:

Either:

---

*Conversion factor method*

Multiply by a unit conversion factor that = 1 (so the measurement itself does not get bigger or smaller) but which includes the old and new units (so the new units appear and new units disappear).

**Example**

A brain surgeon wants to fill his aeroplane with petrol. He needs 60 gallons but has to buy it in litres. How many litres does he need?

We remember that 1 gallon = 4.545 litres so $\dfrac{4.545 \text{ litres}}{1 \text{ gallon}}$ is a fraction that = 1, we can

multiply by it without changing the volume (or divide by $\dfrac{1 \text{ gallon}}{4.545 \text{ litres}}$ ).

We calculate $60 \text{ gallons} = 60 \text{ gallons} \times \dfrac{4.545 \text{ litres}}{1 \text{ gallon}} = 272.7 \text{ litres}$ (note how the "gallons"

units on the top and bottom cancel out, leaving litres as the units).

---

Or

---

*Scaling method*

$60 \text{ gallons} = 60 \times 1 \text{ gallon} = 60 \times 4.545 \text{ litres} = 272.7 \text{ litres}$

---

***What happens if you get it wrong!***

## Metric area and volume conversions

- 1 $m^2$ (a square with 1 m sides) is the same area as a square with 100 cm sides (area 10000 $cm^2$) so to convert square metres ($m^2$) to square cm ($cm^2$) we <u>multiply by 10000</u>.

- 1 $m^3$ (a cube with 1 m sides) has the same volume as a cube with 100 cm sides (volume 1000000 $cm^3$) so to convert cubic metres to cubic cm we <u>multiply by 1000000</u>.

- <u>Need to know:</u>   1 litre = 1000 $cm^3$ = 1.76 pints.

# *Speed, distance and time*

$$S = \frac{D}{T}, \quad ST = D, \quad T = \frac{D}{S}$$
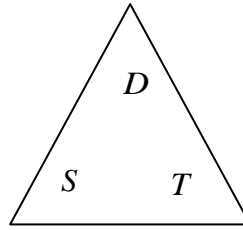


**Examples (check these on your calculator!)**

- I jog 3 miles in 24 minutes, what is my average speed in (a) miles/hour, (b) metres/second

    (a) $\quad S = \dfrac{D}{T} = \dfrac{3 \text{ miles}}{\left(\dfrac{24}{60}\right) \text{ hour}} = 7.5 \text{ miles per hour}$

    (b) 1 mile = 1600 metres so

    $$S = \frac{D}{T} = \frac{3 \times 1600 \text{ metres}}{24 \times 60 \text{ seconds}} = 3\frac{1}{3} \text{ m/s}$$

- I swim for 10 minutes at 0.7 metres per second, how far do I get?

    $$ST = D$$

    10 minutes = 600 seconds,

    0.7 m/s ×600 seconds = 0.7×600 metres = 420 metres.

- I want to cycle 25 miles and can cycle at 15 miles/hour, how long will it take me?

    $$T = \frac{D}{S} = \frac{25 \text{ miles}}{15 \text{ miles/hour}} = \frac{25}{15} \text{ hours} = 1\frac{2}{3} \text{ hours} = 1 \text{ hour 40 minutes}$$

# *Density (using volume to find mass).*

The "density" of a material is the mass of a 1 m$^3$ volume of it.

eg. **(1)** The density of cast iron is 7800 kg/m$^3$.    A sailing yacht has ballast weights of volume $\dfrac{1}{100}$ m$^3$.    How much does each weight weigh?

Mass = $0.01 \times 7800 = 78$ kg

Some typical densities (for interest):

| Steel | 7840 kg/m$^3$ = 7.84 kg/litre |
|---|---|
| Aluminium | 2700 kg/m$^3$ = 2.7 kg/litre |
| Water | 1000 kg/m$^3$ = 1 kg/litre |
| Air | 1.3 kg/m$^3$ = 0.0013 kg/litre |

**(2)**  People in the UK eat 400000 tonnes of chocolate per annum.  If chocolate has density 1333 kg/cubic m, what volume is this?

Mass = volume * density, volume = mass/density = 300 cubic metres.

# Chapter 4: Probability and statistics

## *What is a probability?*

### Definitions

Probability means "what proportion of the time do we expect a particular result from a random process". It is given a number between 0 (never happens) and 1 (certain to always happen).

If we add the probability of every possible result, the total probability must be 1. (There are no other alternatives).

Individual random events are unpredictable. I cannot predict tonight's lottery numbers. What I can predict, though, is the average of a large number of experiments. (e.g. for the lottery I would predict that each number is equally likely – and tables of thousands of past winning numbers show this is true).

### Example

Throwing a dice, the probability of a 6 (if it is a perfectly fair dice) is $\frac{1}{6}$ .

This means we expect a six roughly one throw in every six throws, about 2 sizes in 12 throws, about 1000 sixes in 6000 throws.

[Aside: If we did a huge number of throws, the fraction giving a 6 would actually be very close to $\frac{1}{6}$ .

For example I have just thrown a die 6000 times and I got 1029 sixes. This fraction $\frac{1029}{6000}$ is called a "relative frequency" and it is an experimental estimate of the probability (not exact!)].

### Example

The probability of a student passing GCSE Maths is 0.6

What is the probability of them failing?

 ➢ Since they either pass or fail (no other options), P(fail) = 1 – P(pass) = 1 – 0.6 = 0.4

## Calculating probabilities for simple experiments

We can calculate an *exact* probability in situations where there are several possibilities that are <u>all equally likely</u>. The probability P is then:

$$P = \frac{\text{number of possible "successes"}}{\text{number of possible equally likely results}}$$

**Example**

A bag contains 20 raffle tickets numbered 1 to 20.

If I pick one at random, what is the probability that its number is <6?

➢ A "success" would be a number 1, 2, 3, 4 or 5.

$$P = \frac{\text{number of possible "successes"}}{\text{number of possible equally likely results}} = \frac{5}{20} = \frac{1}{4}$$

## Adding probabilities

We can add probabilities for events that <u>*can never happen together*</u>.

**Example**

The same raffle ticket example: all numbers are equally likely, so the probability of each number being picked is $\frac{1}{20}$ .

The probability of a 1 or 2 or 3 or 4 or 5 is then $\frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} = \frac{5}{20} = \frac{1}{4}$

**Example**

The probability that the next car leaving the car park will be red is 0.3. We can write this as;

P(is red) = 0.3

Given that P(is green) = 0.1, what is the probability that the next car will be <u>red or green</u>?

➢ P(is red or green) = 0.3 + 0.1 = 0.4

(since a car cannot be red and green at the same time).

**Example**

The probability of getting >60% in an exam is 0.5
The probability of getting between 50 and 60% is 0.2
What is the probability of getting >50%?

- ➢ Since all marks over 50 are either in the 50 to 60 range, or over 60, we can add the probabilities:

  P(>50%) = 0.5 + 0.2 = 0.7

## Multiplying probabilities

If two different things *A* and *B* can happen together, and they do not influence each other (they are "independent"), the probability that they do happen together is found by multiplying their individual probabilities.

$$P(A \text{ and } B) = P(A) \times P(B)$$

**Example**

In any month, the probability that I will see a Barn Owl is 0.1
The probability that I will see a Vulcan bomber is 0.5
What is the probability that I see both next month?

- ➢ These events are independent so I can multiply the probabilities.

- ➢ P(Barn Owl & Vulcan bomber) = 0.1 ×0.5 = 0.05

## Summary

- Multiply probability by number of experiments to estimate likely number of successes

- Subtract from 1 to find the only unknown probability

- Add probabilities of A and B (if cannot happen together) to get P(both).

- Multiply to get P(A and B together).

# *What is a statistic?*

A statistic is a number that we calculate from some measurements as a simple way of describing them. The mean, median, mode and range are all statistics.

"Some measurements" will be either given as

- A list of several numbers, eg 1, 3, 4, 9

- A "frequency table" of values

- A frequency table with the values <u>grouped into classes</u>

## Basic definitions

The **mean** is found by adding up all the values and dividing by how many of them there are.

$$\text{mean} = \frac{\text{sum of values}}{\text{number of measurements}}$$

<u>Example</u>

5 dogs with weights 11 kg, 20 kg, 30kg, 17 kg and 62 kg, find the mean weight.

We add the weights:

Total weight = 11 + 20 + 30 + 17 + 62 = 140 kg

Then

$$\text{mean weight} = \frac{140 \text{ kg}}{5} = 28 \text{ kg}$$

The **median** is the <u>central value</u> in a sorted list or in a stem and leaf diagram. So I can sort these dogs by weight, smallest to largest:

11, 17, 20, 30, 62

And I see the median is 20 kg.

[If there had been an even number of dogs, I would take the median to be halfway between the two central values. For instance, adding a sixth dog weighing 41 kg, the sorted list becomes

11, 17, 20, 30, 41, 62

The median is now halfway between 20 and 30, we think $\frac{20+30}{2} = \frac{50}{2} = 25 \text{ kg}$. ]

- For measurements with a few very high or low values, the mean is affected by the extreme values (the huge 62kg dog above) but the median is not, so the median gives a good indication of a typical value.

- But – we cannot use the median for calculations because it ignores the actual value of most data points.  If we want to calculate (for instance) a total weight for the pack of dogs, we have to use the mean.

The **mode** is the most common value, so in a list 1, 1, 1, 2, 2, 5, 5, 5, 5, 9 the mode would be 5 (we have four of them).  The mode is really only useful when we have many items in each category.

For instance, if a car park contains 400 green cars, 500 red cars and 120 blue cars we can say the mode (or "modal colour") is red.

- We can use the mode if the measurements are not numbers (eg the colours red, green blue).

The **range** is the difference between the smallest and largest value.  So for the dogs with weights 11, 17, 20, 30, 62 the range is 62-11 = 51 kg.

## Getting statistics from a frequency table

A list where some numbers are repeated many times eg 1, 1, 1, 2, 2, 2, 2, 5, 5, 9 is usually written as a frequency table.  For instance, if these numbers were for a class of 10 pupils and described how many GCSE passes each pupil had, we could write:

| Score "x" (number of GCSEs passed) | Frequency "f" (number of pupils with this score) | number of GCSEs belonging to people with this score = $f \times x$ |
|---|---|---|
| 1 | 3 | 1x3 = 3 |
| 2 | 4 | 2x4 = 8 |
| 5 | 2 | 5x2 = 10 |
| 9 | 1 | 9x1 = 9 |
| total: | 3+2+4+1 = 10 pupils | total number of GCSEs passed = 3+8+10+9 = 30 |

$$\text{mean number of GCSEs per pupil} = \frac{\text{total number of GCSEs}}{\text{number of pupils}} = \frac{30}{10} = 3$$

(This is just equivalent to doing $\dfrac{1+1+1+2+2+2+2+5+5+9}{10} = \dfrac{30}{10} = 3$)

The mode will be 2 GCSEs per pupil (the most common result).

The median will be halfway between the 5<sup>th</sup> and 6<sup>th</sup> scores in the sorted list, so the median score is also 2.

## Grouped continuous data

Often our measured data can take any value so although we have hundreds of measurements, no two are the same.   We then classify them into groups or "classes" that contain a range of values. Once we have done this, the exact original values are forgotten and we assume that they have the group mid-point value.

For instance, the weights of a team could be tabulated as

| Weight class | Mid-point "x", kg | Frequency "f" | $f \times x$ |
|---|---|---|---|
| $50 \leq x < 60$ kg | 55 | 5 | 5 x 55 = 275 |
| $60 \leq x < 70$ kg | 65 | 8 | 8 x 65 = 520 |
| $70 \leq x < 80$ kg | 75 | 7 | 7 x 75 = 525 |
| | totals: | 20 | 1320 |

The mean weight is then $\dfrac{1320}{20} = 66$ kg.

## Using an "assumed mean"

(also known as "coding")

In any calculation of the mean we can subtract a number from each data value, find the mean, then add it back on.

For instance:
To find the mean of 2001, 2010, 2019:

- subtract 2000,
- find the mean of 1, 10 and 19 ( 1+10+19 = 30,   30÷3 = 10),
- now add 2000 back on to give a final mean 10 + 2000 = 2010.

# Chapter 5:  Using a calculator

You should have your own calculator so you can practice using it at home until you are completely familiar with it.   We sell the Logik LK-185 for £4.50 but most other calculators (e.g. Casio fx-83, fx-991) are also suitable provided they have the [$x^2$], [$x^3$] or [^], [$\sqrt{\ }$] and [ANS] buttons.



General principles:

(1)  You type an expression, then press "=" to make the calculator do it, for instance  2x3+1=

(2)  To enter a number in standard form, use the EXP button, for instance 7.8 [EXP] 3

(3)  To square a number, press [$X^2$] after it, for instance 5 [$X^2$] =

(4)  Higher powers can be found using the [^] button, for instance 7 [^] 5 = would find $7^5$.

(5)  To find a square root, press [$\sqrt{\ }$] before the number, for example  [$\sqrt{\ }$] 256 =

(6)  Use brackets if implied by the expression.

- For instance, to find $\dfrac{10+20}{6}$ you remember that 10 + 20 ÷ 6 would divide 20 by 6, then add the answer to 10 (BIDMAS).  You want both the 10 and 20 to be divided by 6, so we type  (10 + 20) ÷ 6 =

- Similarly to find $\sqrt{100+44}$ you would not want the calculator to do $\sqrt{100}+44$. The top of the square root covers both the 100 and the 44, so these have to be added first, then square rooted. You need brackets  [√] ( 100 + 44 ) =

(7) You can enter a fraction or mixed number using the [ab/c] button.

- To enter $\dfrac{2}{3}$ , press 2 [ab/c] 3.  The display will show 2⌐ 3

- To enter $1\dfrac{2}{3}$ , press  1 [ab/c] 2 [ab/c] 3.  The display will show 1⌐ 2⌐ 3

- You can switch between mixed number and decimal display format by pressing the [ab/c] button repeatedly when an answer is displayed (eg 1⌐ 1⌐ 2 will change to 1.5 and back again).  Press [shift[ [b/c] to convert it into an improper fraction 3⌐ 2

**Beware!**

(1) Squaring negative numbers.  Most calculators obey BIDMAS so if you enter

-1  [$X^2$]=

it will square (an index operation) before thinking about the minus sign.  Effectively it is doing $0-1^2$.

To avoid this, either use brackets ( -1 ) [$X^2$] =  or remember that your answer will be positive anyway, so we can ignore the "-" and just type    1 [$X^2$] =

(2) Dividing by a fraction.

To calculate $\dfrac{30}{\left(\dfrac{3}{2}\right)}$ , either type 30 ÷ (3 ÷ 2 ) =  or 30 ÷ 3⌐ 2

To calculate $\dfrac{\left(\dfrac{30}{3}\right)}{2}$ , type ( 30 ÷ 3 ) ÷ 2 = ,   30 ÷ 3 ÷ 2 =   or just  30⌐ 3 ÷ 2  =

(3)  The [ab/c] button enters a number into the calculator.  ***Be very careful*** if you use this button! I prefer to avoid it.

- Indices are performed **before** the number is entered so we can enter $1\dfrac{1}{2^3}$ by

  pressing 1 [ab/c] 1 [ab/c] 2 [$X^3$] =

- +, -, x, ÷ operate **after** the fraction is entered so 1 [ab/c] 1 [ab/c] 3 x 2 =  will give us

$$1\frac{1}{3} \times 2 = 2\frac{2}{3} \text{ , not } 1\frac{1}{3 \times 2} = 1\frac{1}{6}$$

**(4)** The % button (above [=]) does not do what you expect!  It will give totally peculiar answers. **Do not use it!**

**The ANS button**

This lets you use the answer from one calculation in a new calculation.  For instance, to calculate $\sqrt{3^2 + 4^2}$ I could type  [√] (3 [X²] + 4 [X²] ) = but it is easier to do the adding first (so I do not need brackets):

3 [X²] + 4 [X²] =

and then square root the answer [√] [ANS] =

**Using ANS to repeat a calculation.**

You must have the calculator in COMP mode.

- Type a number, press =.  This saves it as "ANS"

- Type an expression including ANS, press =

Now you can put in a new number and use the "up" key to go back to the previous calculation.  Press "=" **once** and it will repeat the calculation with your new ANS value.

For instance, to fill in the y values in a table for $y = x^2 - 5x + 2$

| x | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| y |  |  |  |  |  |

-2 =                              (your first x value)

ANS [x²] – 5 x ANS + 2 =          (answer y = 16, put into your table)

-1 =                              (now -1 is "ANS")

Press up arrow, then = again.     (answer 8, put into the table)

0 =                               (now 0 is "ANS")

Press up arrow, then = again.     (answer 2, put into the table)

**Percentage calculations**

To find 14% of 2000:

$14 \div 100 \times 2000 =$

To find £500 increased by 12%

$500 \times (1 + 12 \div 100) =$

(or $500 \times 1.12 =$ )

To find 60 out of 3000 as a percentage

$60 \div 3000 \times 100 =$

A car has increased in price by 20% and is now £6000.  What was it originally?

$6000 \div 1.2 =$

or $6000 \div (1 + 20 \div 100) =$